dropped down a 100-meter-deep mine shaft at the equator, and let us find the total deflection by the time it hits the bottom. The time to reach the bottom is determined by the last of equations (9.55) as $t = \sqrt{2h/g}$, and (9.57) gives for the total easterly deflection (putting $\theta = 90°$ and $g \approx 10$ m/s^2)

$$x = \frac{1}{3}\Omega g \left(\frac{2h}{g}\right)^{3/2}$$

$$\approx \frac{1}{3} \times (7.3 \times 10^{-5}\,\text{s}^{-1}) \times (10\,\text{m/s}^2) \times (20\,\text{s}^2)^{3/2} \approx 2.2\,\text{cm}$$

a small deflection, but certainly detectable. A small easterly deflection of this type was actually predicted by Newton and verified by his rival Robert Hooke (of Hooke's law fame, 1635–1703), although it was not properly explained until the Coriolis effect was understood.

9.9 The Foucault Pendulum

As a final and striking application of the Coriolis effect, let us consider the Foucault pendulum, which can be seen in many science museums around the world and is named for its inventor, the French physicist Jean Foucault (1819–1868). This is a pendulum made of a very heavy mass m suspended by a light wire from a tall ceiling. This arrangement allows the pendulum to swing freely for a very long time and to move in both the east–west and north–south directions. As seen in an inertial frame, there are just two forces on the bob, the tension $\mathbf{T}$ in the wire and the weight $m\mathbf{g}_o$. In the rotating frame of the earth, there are also the centrifugal and Coriolis forces, so the equation of motion in the earth's frame is

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g}_o + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}.$$

Exactly as in the previous section, the second and third terms on the right combine to give $m\mathbf{g}$, where $\mathbf{g}$ is the observed free-fall acceleration, and the equation of motion becomes

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}. \tag{9.58}$$

We can now choose our axes as in the previous section, so that x is east, y is north, and z vertically up (direction of $-\mathbf{g}$), and the pendulum is as shown in Figure 9.16.

I shall restrict our discussion to the case of small oscillations, so that the angle β between the pendulum and the vertical is always small. This allows two simplifying approximations: First, the z component of the tension $\mathbf{T}$ is well approximated by the magnitude; that is, $T_z = T\cos\beta \approx T$. Second, it is not hard to see that, for small oscillations, $T_z \approx mg$.[14] Putting these two approximations together, we can write

$$T \approx mg. \tag{9.59}$$

[14] Look at the z component of (9.58). In the limit of small oscillations, the term on the left and the last term on the right both approach zero, and you're left with $T_z - mg = 0$.

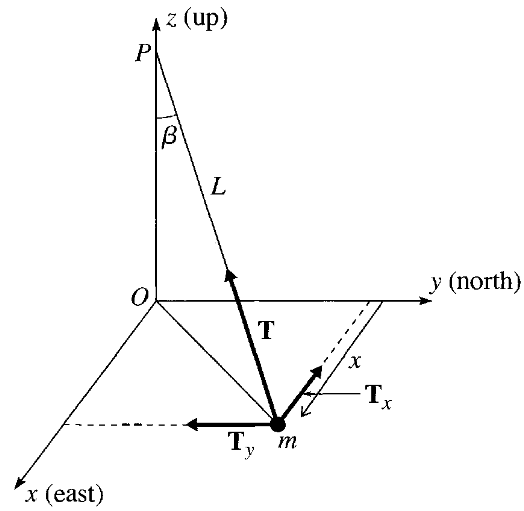

Figure 9.16 A Foucault pendulum comprises a bob of mass m suspended by a light wire of length L from the point P on a high ceiling. The tension force on the bob is shown as $\mathbf{T}$ and its x and y components are T_x and T_y. For small oscillations the angle β is very small.

We now need to examine the x and y components of the equation of motion (9.58). This requires that we identify the x and y components of $\mathbf{T}$. If you look at Figure 9.16, you will see that, by similar triangles, $T_x/T = -x/L$ and similarly for T_y. Combining this with (9.59), we find that

$$T_x = -mgx/L \quad \text{and} \quad T_y = -mgy/L. \tag{9.60}$$

The x and y components of $\mathbf{g}$ are, of course, zero, and the components of $\dot{\mathbf{r}} \times \mathbf{\Omega}$ are given in (9.52). Putting all of these into (9.58), we find (after canceling a factor of m and dropping a term involving $\dot{z}$, which is negligible compared to $\dot{x}$ or $\dot{y}$ for small oscillations)

$$\left.\begin{aligned} \ddot{x} &= -gx/L + 2\dot{y}\Omega\cos\theta \\ \ddot{y} &= -gy/L - 2\dot{x}\Omega\cos\theta. \end{aligned}\right\} \tag{9.61}$$

where as usual θ denotes the colatitude of the location of the experiment. The factor g/L is just ω_o^2, where ω_o is the natural frequency of the pendulum, and $\Omega\cos\theta$ is just Ω_z, the z component of the earth's angular velocity. Thus these two equations of motion can be rewritten as

$$\left.\begin{aligned} \ddot{x} - 2\Omega_z\dot{y} + \omega_o^2 x &= 0 \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_o^2 y &= 0. \end{aligned}\right\} \tag{9.62}$$

We can solve the coupled equations (9.62) using the trick, introduced in Chapter 2, of defining a complex number

$$\eta = x + iy.$$

Recall that not only does this complex number contain the same information as the position in the xy plane, but a plot of η in the complex plane is an actual bird's eye view of the pendulum's projected position (x, y). If we multiply the second equation of (9.62) by i and add it to the first, we get the single differential equation

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_o^2\eta = 0. \tag{9.63}$$

This is a second-order, linear, homogeneous differential equation and so has exactly two independent solutions. Thus if we can find two independent solutions, we shall know that the most general solution is a linear combination of these two. As often happens, we can find two independent solutions by inspired guesswork: We guess that there is a solution of the form

$$\eta(t) = e^{-i\alpha t} \tag{9.64}$$

for some constant α. Substituting this guess into (9.63), we see immediately that it is a solution if and only if α satisfies

$$\alpha^2 - 2\Omega_z\alpha - \omega_o^2 = 0$$

or

$$\begin{aligned}\alpha &= \Omega_z \pm \sqrt{\Omega_z^{\,2} + \omega_o^2} \\ &\approx \Omega_z \pm \omega_o\end{aligned} \tag{9.65}$$

where the last line is an extremely good approximation since the earth's angular velocity Ω is so very much smaller than the pendulum's ω_o. This gives us the required two independent solutions, and the general solution to the equation of motion (9.63) is

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}\right). \tag{9.66}$$

To see what this solution looks like, we need to fix the two constants C_1 and C_2 by specifying the initial conditions. Let us suppose that at $t = 0$ the pendulum has been pulled aside in the x direction (east) to a position $x = A$ and $y = 0$, and is released from rest ($v_{xo} = v_{yo} = 0$). With these initial conditions, you can easily check that[15] $C_1 = C_2 = A/2$, and our solution becomes

$$\eta(t) \equiv x(t) + iy(t) = Ae^{-i\Omega_z t}\cos\omega_o t. \tag{9.67}$$

At $t = 0$ the complex exponential is equal to one, and $x = A$, while $y = 0$. Because $\Omega_z \ll \omega_o$, the cosine factor in (9.67) makes many oscillations before the exponential changes appreciably from one. This implies that, initially, $x(t)$ oscillates with angular frequency ω_o between $\pm A$, while y remains close to zero. That is, initially, the pendulum swings in simple harmonic motion along the x axis, as indicated in Figure 9.17(a).

[15] Actually, there is a small subtlety, in that these simple values depend on the (true) assumption that $\Omega_z \ll \omega_o$, as you will see when you check them.

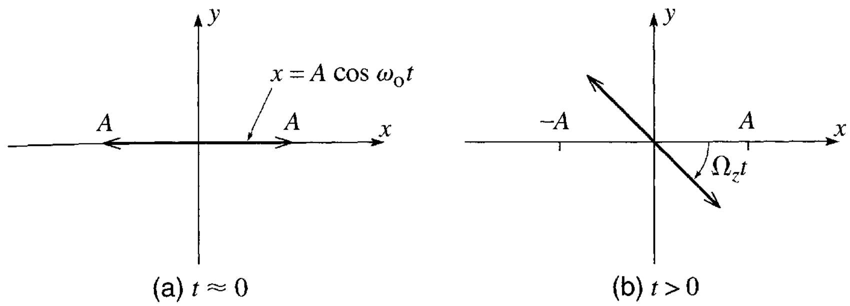

Figure 9.17 Overhead views of the motion of a Foucault pendulum. **(a)** For a while after being released, the pendulum swings back and forth along the x axis, with amplitude A and frequency ω_0. **(b)** As time advances, the plane of its oscillations slowly rotates with angular velocity equal to Ω_z, the z component of the earth's angular velocity.

However, eventually the complex exponential $e^{-i\Omega_z t}$ begins to change, causing the complex number $\eta = x + iy$ to rotate through an angle $\Omega_z t$. In the Northern Hemisphere, where Ω_z is positive, this means that the number $x + iy$ continues to oscillate sinusoidally (due to the factor $\cos\omega_0 t$), but in a direction that rotates clockwise. That is, the plane in which the pendulum is swinging rotates slowly clockwise, with angular velocity Ω_z, as indicated in Figure 9.17(b). In the Southern Hemisphere, where Ω_z is negative, the corresponding rotation is counterclockwise.

If the Foucault pendulum is located at colatitude θ (latitude $90° - \theta$), then the rate at which its plane of oscillation rotates is

$$\Omega_z = \Omega\cos\theta. \tag{9.68}$$

At the North Pole ($\theta = 0$), $\Omega_z = \Omega$ and the rate of rotation of the pendulum is the same as the earth's angular velocity. This result is easy to understand: As seen in an inertial (nonrotating) frame, a Foucault pendulum at the North Pole would obviously swing in a fixed plane; meanwhile, as seen in the same inertial frame, the earth is rotating counterclockwise (as seen from above) with angular velocity Ω. Clearly then, as seen from the earth, the pendulum's plane of oscillation has to be rotating clockwise with angular velocity Ω.

At any other latitude, the result is much more complicated from an inertial point of view, but the rate of rotation of the Foucault pendulum is easily calculated from (9.68). At the equator ($\theta = 90°$), $\Omega_z = 0$ and the pendulum does not rotate. At a latitude around 42° (the approximate latitude of Boston, Chicago, or Rome),

$$\Omega_z = \Omega\cos 48° \approx \tfrac{2}{3}\Omega.$$

Since Ω equals 360°/day, $\frac{2}{3}\Omega = 240°/\text{day}$, and we see that in the course of 6 hours (a time for which a long, well-built pendulum will certainly continue to swing without significant damping), the pendulum's plane of motion will rotate through 60° — an easily observable effect.