Problem 1 :

1. The matrix A=

$1/\sqrt{2}$	$1/\sqrt{2}$	0
0	0	1
$1/\sqrt{2}$	$-1/\sqrt{2}$	0

represents a finite rotation about a certain axis. Find the direction cosines of the axis and the angle of rotation.

In[ ]:= $m = \left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \{0, 0, 1\}, \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right\}\right\};$

```
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[ ]:=
```
Det[m]
```

Out[ ]= 1

In[ ]:=
```
eval = Eigenvalues [m] // N
```

Out[ ]= {-0.146447 + 0.989219 ⅈ, -0.146447 - 0.989219 ⅈ, 1.}

In[ ]:=
```
eval[[1]] // Abs
```

Out[ ]= 1.

In[ ]:=
```
sol = Solve[eval[[1]] == Exp[I ϕ], ϕ][[1]] /. C[1] → 0 // Chop
```

••• Solve : Inverse functions are being used by Solve , so some solutions may not be found ; use Reduce for complete solution information .

Out[ ]= {ϕ → 1.71777}

In[ ]:= $\frac{\phi}{\pi}$ /. sol

Out[ ]= 0.546784

In[ ]:=
```
evec = Eigenvectors [m] // FullSimplify
```

Out[ ]= {{-0.354 … + 0.410 … ⅈ, -0.146 … - 0.989 … ⅈ, 1},
{-0.354 … - 0.410 … ⅈ, -0.146 … + 0.989 … ⅈ, 1}, $\{1 + \sqrt{2}, 1, 1\}$}

Problem 3:

3. The matrix L represents a rotation by an angle ϕ around some axis. The eigenvalues of L are λ1 = +1, λ2 =(√3+i)/2 . λ3 = (√3-i)/2. Find the angle ϕ.

In[ ]:=
```
Clear["Global`*"]
```

In[ ]:=
```
m = {{Cos[θ], Sin[θ], 0}, {-Sin[θ], Cos[θ], 0}, {0, 0, 1}};
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] & 0 \\ -\text{Sin}[\theta] & \text{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
ev = Eigenvalues[m] // TrigToExp
```

Out[ ]= $\{1, e^{-i\theta}, e^{i\theta}\}$

In[ ]:= $\text{ev2} = \left\{1, \frac{\sqrt{3} + \text{I}}{2}, \frac{\sqrt{3} - \text{I}}{2}\right\};$

In[ ]:= $\text{Solve}\left[\text{Exp}[\text{I}\ \theta] == \frac{\sqrt{3} + \text{I}}{2}, \theta\right] /. \{\text{C}[1] \to 0\}$ // Simplify

Out[ ]= $\left\{\left\{\theta \to \frac{\pi}{6}\right\}\right\}$

Problem 5:

5. We want to demonstrate that the 3x3 and 2x2 representation of rotations are identical for infinitesimal rotations on a vector v={x,y,z}.
The 3x3 rotations can be represented as Exp[$\theta\, n \bullet M$] where n is the axis vector, and M are the matrix generators of the rotation, and the rotated vector is Exp[$\theta\, n \bullet M$]•v.
The 2X2 rotations can be represented as Exp[-I $\theta/2$ $n \bullet \sigma$] where n is the axis vector, and σ are the matrix generators of the rotation, and the rotated vector is Exp[-I $\theta/2$ $n \bullet \sigma$]•v.
a) Perform an infinitesimal rotation about the x-axis with both representations, and show the components transform identically.
b) Repeat for the y-axis.
c) Repeat for the z-axis.

3 - D rotations :

In[ ]:=
```
m0 = DiagonalMatrix [{1, 1, 1}];
m1 = {{0, 0, 0}, {0, 0, -1}, {0, 1, 0}};
m2 = {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}};
m3 = {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}};
MatrixForm /@ {m0, m1, m2, m3}
```

Out[ ]=
$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right\}$$

In[ ]:=
```
series = Series[Exp[-I θ m] - 1 , {θ, 0, 2}] // Normal
```

Out[ ]=
$$-i\, m\, \theta - \frac{m^2\, \theta^2}{2}$$

In[ ]:=
```
tmp1 = series /. {m^2 → m1.m1, m → m1};
tmp1 // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix}0&0&0\\0&\frac{\theta^2}{2}&i\,\theta\\0&-i\,\theta&\frac{\theta^2}{2}\end{pmatrix}$$

In[ ]:=
```
tmp2 = tmp1 + m0;
tmp2 // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix}1&0&0\\0&1+\frac{\theta^2}{2}&i\,\theta\\0&-i\,\theta&1+\frac{\theta^2}{2}\end{pmatrix}$$

In[ ]:=
```
rotx = tmp2 /. {1 + θ^2/2 → Cos[θ], i θ → Sin[θ], -i θ → -Sin[θ]};
rotx // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix}1&0&0\\0&\text{Cos}[\theta]&\text{Sin}[\theta]\\0&-\text{Sin}[\theta]&\text{Cos}[\theta]\end{pmatrix}$$

3 - D rotations : commutator

In[ ]:=
```
comm[x_, y_] := x.y - y.x
```

In[ ]:= comm[m1, m2] - m3

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[ ]:= comm[m2, m3] - m1

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[ ]:= comm[m3, m1] - m2

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

3 - D rotations : Rotate about x-axis:

In[ ]:=
```
vec = {x, y, z};
rotx.vec
```

Out[ ]= {x, y Cos[θ] + z Sin[θ], z Cos[θ] - y Sin[θ]}

In[ ]:= Series[rotx.vec, {θ, 0, 1}] // Normal

Out[ ]= {x, y + z θ, z - y θ}

---

3 - D Rotations: 2x2

In[ ]:=
```
s0 = {{1, 0}, {0, 1}};
s1 = {{0, 1}, {1, 0}};
s2 = {{0, -I}, {I, 0}};
s3 = {{1, 0}, {0, -1}};
MatrixForm /@ {s0, s1, s2, s3}
```

Out[ ]= $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$

2 - D rotations : generator

In[ ]:= tmp1 = Series[Exp[+ I θ / 2 s] - 1 , {θ, 0, 2}] // Normal

Out[ ]= $\frac{i\, s\, \theta}{2} - \frac{s^2\, \theta^2}{8}$

In[ ]:=
```
tmp2 = tmp1 /. {s^2 → s1.s1, s → s1};
tmp3 = s0 + tmp2;
tmp3 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 - \frac{\theta^2}{8} & \frac{i\,\theta}{2} \\ \frac{i\,\theta}{2} & 1 - \frac{\theta^2}{8} \end{pmatrix}$$

In[ ]:=
$$\text{rules} = \left\{\theta^2 \to 8\left(1 - \text{Cos}\left[\frac{\theta}{2}\right]\right),\ \theta \to 2\,\text{Sin}\left[\frac{\theta}{2}\right]\right\};$$

In[ ]:=
```
srotx = tmp3 /. rules;
srotx // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

2 - D rotations : commutator

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s1, s2] - \text{I}\,s3$$

Out[ ]= {{0, 0}, {0, 0}}

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s2, s3] - \text{I}\,s1$$

Out[ ]= {{0, 0}, {0, 0}}

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s3, s1] - \text{I}\,s1$$

Out[ ]= {{0, 1 − i}, {−1 − i, 0}}

3 - D rotations : Rotate about x-axis:

In[ ]:=
```
vec2 = x s1 + y s2 + z s3;
vec2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z & x - i\,y \\ x + i\,y & -z \end{pmatrix}$$

In[ ]:=
```
srotx // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[ ]:=
```
Clear[conjugate]
conjugate[mat_] := mat /. Complex[a_, b_] :→ Complex[a, -b]
```

In[ ]:=
```
srotxCT = srotx // conjugate // Transpose;
srotxCT // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & -i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ -i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[ ]:=
```
output = srotx.vec2.srotxCT // Simplify;
output // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z\,\text{Cos}[\theta] - y\,\text{Sin}[\theta] & x - i\,y\,\text{Cos}[\theta] - i\,z\,\text{Sin}[\theta] \\ x + i\,y\,\text{Cos}[\theta] + i\,z\,\text{Sin}[\theta] & -z\,\text{Cos}[\theta] + y\,\text{Sin}[\theta] \end{pmatrix}$$

In[ ]:=
```
vec2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z & x - i\,y \\ x + i\,y & -z \end{pmatrix}$$

Compare

In[ ]:=
```
vec = {x, y, z};
rotx.vec
```

Out[ ]= $\{x,\ y\,\text{Cos}[\theta] + z\,\text{Sin}[\theta],\ z\,\text{Cos}[\theta] - y\,\text{Sin}[\theta]\}$