Problem 5:

5. We want to demonstrate that the 3x3 and 2x2 representation of rotations are identical for infinitesimal rotations on a vector v={x,y,z}.
The 3x3 rotations can be represented as Exp[$\theta\, n$•M] where n is the axis vector, and M are the matrix generators of the rotation, and the rotated vector is Exp[$\theta\, n$•M]•v.
The 2X2 rotations can be represented as Exp[-I $\theta/2$ n•σ] where n is the axis vector, and σ are the matrix generators of the rotation, and the rotated vector is Exp[-I $\theta/2$ n•σ]•v.
a) Perform an infinitesimal rotation about the x-axis with both representations, and show the components transform identically.
b) Repeat for the y-axis.
c) Repeat for the z-axis.

3 - D rotations :

In[ ]:=
```
m0 = DiagonalMatrix [{1, 1, 1}];
m1 = {{0, 0, 0}, {0, 0, -1}, {0, 1, 0}};
m2 = {{0, 0, 1}, {0, 0, 0}, {-1, 0, 0}};
m3 = {{0, -1, 0}, {1, 0, 0}, {0, 0, 0}};
MatrixForm /@ {m0, m1, m2, m3}
```

Out[ ]=
$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right\}$$

In[ ]:=
```
series = Series[Exp[-I θ m] - 1 , {θ, 0, 2}] // Normal
```

Out[ ]=
$$-i\, m\, \theta - \frac{m^2\, \theta^2}{2}$$

In[ ]:=
```
tmp1 = series /. {m^2 → m1.m1, m → m1};
tmp1 // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix}0&0&0\\0&\frac{\theta^2}{2}&i\,\theta\\0&-i\,\theta&\frac{\theta^2}{2}\end{pmatrix}$$

In[ ]:=
```
tmp2 = tmp1 + m0;
tmp2 // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix}1&0&0\\0&1+\frac{\theta^2}{2}&i\,\theta\\0&-i\,\theta&1+\frac{\theta^2}{2}\end{pmatrix}$$

In[ ]:=
```
rotx = tmp2 /. {1 + θ^2/2 → Cos[θ], ⅈ θ → Sin[θ], -ⅈ θ → -Sin[θ]};
rotx // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\theta] & \text{Sin}[\theta] \\ 0 & -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

3 - D rotations : commutator

In[ ]:=
```
comm[x_, y_] := x.y - y.x
```

In[ ]:=
```
comm[m1, m2] - m3
```

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[ ]:=
```
comm[m2, m3] - m1
```

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

In[ ]:=
```
comm[m3, m1] - m2
```

Out[ ]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

3 - D rotations : Rotate about x-axis:

In[ ]:=
```
vec = {x, y, z};
rotx.vec
```

Out[ ]= {x, y Cos[θ] + z Sin[θ], z Cos[θ] - y Sin[θ]}

In[ ]:=
```
Series[rotx.vec, {θ, 0, 1}] // Normal
```

Out[ ]= {x, y + z θ, z - y θ}

---

3 - D Rotations: 2x2

In[ ]:=
```
s0 = {{1, 0}, {0, 1}};
s1 = {{0, 1}, {1, 0}};
s2 = {{0, -I}, {I, 0}};
s3 = {{1, 0}, {0, -1}};
MatrixForm /@ {s0, s1, s2, s3}
```

Out[ ]=

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

2 - D rotations : generator

In[ ]:=
```
tmp1 = Series[Exp[+ I θ / 2 s] - 1 , {θ, 0, 2}] // Normal
```

Out[ ]=
$$\frac{i\, s\, \theta}{2} - \frac{s^2\, \theta^2}{8}$$

In[ ]:=
```
tmp2 = tmp1 /. {s^2 → s1.s1, s → s1};
tmp3 = s0 + tmp2;
tmp3 // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 - \frac{\theta^2}{8} & \frac{i\,\theta}{2} \\ \frac{i\,\theta}{2} & 1 - \frac{\theta^2}{8} \end{pmatrix}$$

In[ ]:=
$$\text{rules} = \left\{\theta^2 \to 8\left(1 - \text{Cos}\left[\frac{\theta}{2}\right]\right),\ \theta \to 2\,\text{Sin}\left[\frac{\theta}{2}\right]\right\};$$

In[ ]:=
```
srotx = tmp3 /. rules;
srotx // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

2 - D rotations : commutator

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s1, s2] - I\, s3$$

Out[ ]= {{0, 0}, {0, 0}}

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s2, s3] - I\, s1$$

Out[ ]= {{0, 0}, {0, 0}}

In[ ]:=
$$\frac{1}{2}\,\text{comm}[s3, s1] - I\, s1$$

Out[ ]= {{0, 1 - i}, {-1 - i, 0}}

3 - D rotations : Rotate about x-axis:

In[ ]:=
```
vec2 = x s1 + y s2 + z s3;
vec2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z & x - i\, y \\ x + i\, y & -z \end{pmatrix}$$

In[ ]:=
```
srotx // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[ ]:=
```
Clear[conjugate]
conjugate[mat_] := mat /. Complex[a_, b_] :→ Complex[a, -b]
```

In[ ]:=
```
srotxCT = srotx // conjugate // Transpose;
srotxCT // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\theta}{2}\right] & -i\,\text{Sin}\left[\frac{\theta}{2}\right] \\ -i\,\text{Sin}\left[\frac{\theta}{2}\right] & \text{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[ ]:=
```
output = srotx.vec2.srotxCT // Simplify;
output // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z\,\text{Cos}[\theta] - y\,\text{Sin}[\theta] & x - i\, y\,\text{Cos}[\theta] - i\, z\,\text{Sin}[\theta] \\ x + i\, y\,\text{Cos}[\theta] + i\, z\,\text{Sin}[\theta] & -z\,\text{Cos}[\theta] + y\,\text{Sin}[\theta] \end{pmatrix}$$

In[ ]:=
```
vec2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} z & x - i\, y \\ x + i\, y & -z \end{pmatrix}$$

Compare

In[ ]:=
```
vec = {x, y, z};
rotx.vec
```

Out[ ]= {x, y Cos[θ] + z Sin[θ], z Cos[θ] - y Sin[θ]}