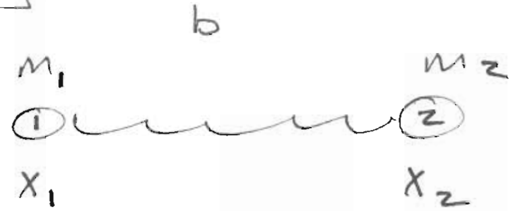
Ch 6



$$V = \frac{1}{2} K \left(x_2 - x_1 - b \right)^2 = \frac{1}{2} K \left(x_2^2 + x_1^2 + b^2 - 2 x_1 x_2 + 2b (x_1 - x_2) \right)$$

$$T = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) m$$

Introduce $\eta_1 = x_1 - x_{10}$ $\qquad x_{20} - x_{10} = b$

$\eta_2 = x_2 - x_{20}$

$$V = \frac{1}{2} K \left(\eta_2 - \eta_1 \right)^2 = \frac{1}{2} K \left(\eta_2^2 + \eta_1^2 - 2\eta_1 \eta_2 \right) = \frac{1}{2} K \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T = \frac{1}{2} m \left(\dot{\eta}_1^2 + \dot{\eta}_2^2 \right)$$

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or $$V(x_1 x_2) = V(x_{10}\, x_{20}) + 0 + \frac{1}{2!} \underbrace{\left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)}_{V_{ij}} (x_i - x_{i0}) \cdot (x_j - x_{j0})$$

$$\frac{\partial^2 V}{\partial x_1^2} = \frac{1}{2} K = \frac{\partial^2 V}{\partial x_2^2}$$

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = -K$$

$\therefore$ $$V_{ij} = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}$$ since $V = V_0 + \frac{1}{2} K \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Then $$V(x_1 x_2) = \frac{1}{2!} V_{ij}\, \eta_i\, \eta_j$$

$\therefore$

$$V = \vec{n} \cdot \frac{\overleftrightarrow{V}}{2} \cdot \vec{n} = \frac{1}{2} K \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$T = \dot{\vec{n}} \cdot \frac{\overleftrightarrow{I}}{2} \cdot \dot{\vec{n}} = \frac{1}{2} m \begin{pmatrix} \dot{n}_1 \\ \dot{n}_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{n}_1 \\ \dot{n}_2 \end{pmatrix}$$

~~Transform V to be diagonal~~

$$L = T - V = \frac{1}{2} T_{ij} \dot{n}_i \dot{n}_j - \frac{1}{2} V_{ij} n_i n_j$$

$$\frac{\partial L}{\partial n_i} = -\frac{1}{2} V_{ij} n_j \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{n}_i} = +\frac{1}{2} T_{ij} \ddot{n}_j$$

$\therefore$ Eq of motion $\Rightarrow$ $\frac{1}{2} \left(T_{ij} \ddot{n}_j + V_{ij} n_j \right) = 0$

IF V_{ij} were diag

$$\Rightarrow \quad T_i \ddot{n}_i + V_i n_j = 0$$

Guess $n_i = C a_i e^{-i\omega t}$ $\qquad \ddot{n}_i = -\omega^2 n_i$

$$\Rightarrow \quad T_i \omega^2 = V_i \quad \Rightarrow \quad \omega = \sqrt{\frac{V_i}{T_i}}$$

Diagonalize V

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = 0 \qquad \lambda(\lambda-2) = 0$$

$$\lambda = 0,\ 2$$

$$v_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$X^+ V X = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\underbrace{\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}}_{\begin{pmatrix}0 & 2\\0 & -2\end{pmatrix}}$$

$$\frac{1}{2}\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}$$

Think of X as a coord Tx $\quad x' = X x$

$\therefore\ V = \vec{x}^{+} \cdot \overleftrightarrow{V}\,\vec{x} = x' \cdot (X^+ V X) \cdot x'$

For $T = \dot{x}^{+} \cdot \overleftrightarrow{T}\,\dot{x} = \dot{x}'^{+}\, T'\, \dot{x}'$

$$T' = X^+ T X = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & \\ & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \mathbb{1}\,T$$

$\therefore$ in primed basis:

$$L = T' - V' = \vec{\dot{x}}' \cdot \overleftrightarrow{\mathbb{1}} \cdot \vec{\dot{x}}' - \vec{x}' \cdot \overleftrightarrow{V}' \cdot \vec{x}'$$

$$= \dot{x}'_i\, \overline{T}_{ii}\, \dot{x}'_i - x_i\, V'_{ii}\, x_i$$

Eq Mot $\Rightarrow$

$$T_i\, \ddot{x}_i + \lambda'_i\, x_i = 0$$

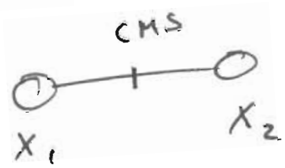
$$\omega^2 = \frac{\lambda_i}{T_i} = \quad \omega_0^2 = 0 \qquad \omega_2^2 = \frac{2\cdot\frac{k}{2}}{m/2} = \frac{2k}{m}$$

$$v_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$\omega = 0$

$\omega = \frac{2k}{m}$

Choose normal coords from start.



Let $\eta_1 = \frac{x_2 + x_1}{2} \qquad x_2 = \eta_1 + \eta_2$

$\eta_2 = \frac{x_2 - x_1}{2} \qquad x_1 = \eta_1 - \eta_2$

$$\therefore \quad V = \frac{1}{2} k\,(x_2 - x_1 - b)^2 = \frac{1}{2} k\,(2\eta_2 - b)^2$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{m}{2} 2\left(\dot{\eta}_1^2 + \dot{\eta}_2^2\right)$$

$$\frac{\partial^2 V}{\partial \eta_2^2} = \frac{1}{2} k \cdot 4 \qquad \frac{1}{2} V_{ij} = \frac{k}{2}\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}$$

$$\frac{1}{2} T_{ij} = \frac{m}{2}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

$$\therefore \quad L = T - V$$

$$\Rightarrow \quad T_{ij}\,\ddot{\eta}_j + V_{ij}\,\eta_j = 0$$

$$\omega^2 = \frac{V_{ii}}{T_{ii}} = 0,\ \frac{4 \cdot k/2}{2 \cdot m/2} = \frac{2k}{m}$$

Driving ~~Dissipative~~ Forces :

$$Q_i = a_{ij} F_j$$

$$\Rightarrow \quad \ddot{x}_i + \omega^2 x_i = Q_i$$

Take $\quad Q = Q_0 e^{i\Omega t}$ driving Force

Guess $\quad x(t) = x_0 e^{i\Omega t}$ at driving Ω

$$\ddot{x} = -\Omega^2 x$$

$$\therefore \quad (\omega^2 - \Omega^2) x_0 e^{i\Omega t} = Q_0 e^{i\Omega t}$$

$$x_0 = \frac{Q_0}{(\omega^2 - \Omega^2)}$$

$$x(t) = x_0 e^{i\Omega t} = \frac{Q_0}{(\omega^2 - \Omega^2)} e^{i\Omega t}$$

Dissipative función :

$$F = \frac{1}{2} F_{ij} \, \mathring{\eta}_i \mathring{\eta}_j \qquad \text{Force}$$

$$Eq \Longrightarrow \qquad T_{ij} \, \ddot{\eta}_j + F_{ij} \, \mathring{\eta}_j + V_{ij} \, \eta_j = 0$$

Tx to normal coords

$$\ddot{\zeta}_i + F_i \, \mathring{\zeta}_i + \omega_i^2 \, \zeta_i = 0$$

$$\text{Guess} \qquad \zeta_i = C_i \, e^{-i \Omega_i t}$$

$$\mathring{\zeta}_i = -i \Omega \, \zeta_i$$

$$\ddot{\zeta}_i = -\Omega^2 \, \zeta_i$$

$$\left(\omega_i^2 - \Omega_i^2 - i \Omega_i F_i \right) \zeta_i = 0$$

$$\cancel{\Omega_i^2 = \omega_i^2 - i \Omega_i F_i}$$

$$\Omega_i^2 + \Omega_i \left(+ i F_i \right) + \left(-\omega_i^2 \right) = 0$$

$$\Omega_i = -i F_i \pm \sqrt{-F_i^2 + 4 \omega_i^2} \; // \, 2$$

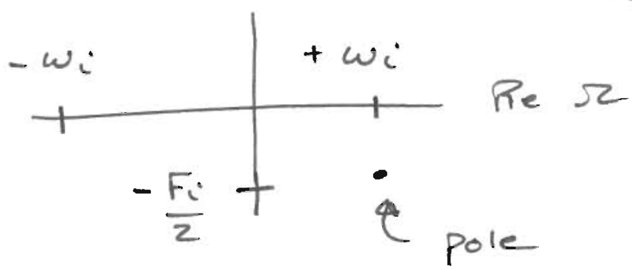
$$= \pm \sqrt{\omega_i^2 - \frac{F_i^2}{4}} - \frac{i F_i}{2}$$

For $\quad F_i \ll \omega_i$

$$\Omega_i \sim \pm\, \omega_i \left(1 - \frac{F_i^2}{8\omega_i^2}\right) - i\,\frac{F_i}{2}$$

$$\sim \pm\left(\omega_i - \frac{F_i^2}{8\omega_i}\right) - i\,\frac{F_i}{2} \;\hat{=}\; \pm\,\omega_i - i\,\frac{F_i}{2}$$

neglect



For $\quad \Omega_i = \pm\,\omega_i - i\,\frac{F_i}{2}$

$$\xi_i = c_i\, e^{-i\Omega_i \cancel{t}} =$$

$$= c_i\, e^{\left(\mp i\,\omega_i - \frac{F_i}{2}\right) t}$$

Oscillation Frequency

damping Factor

$$\xi_i = e^{\pm i t \sqrt{\omega_i^2 - \frac{F_i^2}{4}}} \cdot e^{-\frac{F_i}{2} t}$$

always less
than ω undamped

Dissipative w/ Driving Force:

Ω driving
ω resonant

For norm. coords

$$\ddot{\zeta} + F\dot{\zeta} + \omega^2 \zeta = Q_0 e^{-i\Omega t}$$

For $\zeta = \zeta_0 e^{-i\sigma t}$

$$\dot{\zeta} = -i\sigma\zeta$$
$$\ddot{\zeta} = -\sigma^2\zeta$$

$$(\omega^2 - \sigma^2 - iF\sigma)\zeta_0 e^{-i\sigma t} = Q_0 e^{-i\Omega t} \Rightarrow \sigma = \Omega$$

$$\Omega^2 + \Omega(iF) + \left(\frac{Q_0}{\zeta_0} - \omega^2\right) = 0$$

$$\zeta_0 = \frac{Q_0}{\omega^2 - \Omega^2 - iF\Omega} = \frac{1}{a - ib} \cdot \frac{a+ib}{a+ib} = \frac{a+ib}{a^2-b^2}$$

$$\mathrm{Re}\,\zeta_0 = \frac{Q_0(\omega^2 - \Omega^2)}{(\omega^2-\Omega^2)^2 - (F\Omega)^2}$$

$$\zeta_0 = \frac{Q_0}{(\omega - \omega_0)(\omega - \omega_0^*)} = \frac{Q_0}{\omega^2 - \omega(2\,\mathrm{Re}\,\omega_0)}$$