Overview

1) Warmup : Specify problem
   D'Alemberts princ.
   Lagrange Eq.

2) Lagrange $\Longleftrightarrow$ Hamiltons Princ.
   Calc of variations

3) Kepler Problem
   And variations , scattering theory

4) Rigid Body

5) "

6) Small Oscillations : Pert. theory

7) Relativity

8) Hamilton Eq of motion.
   Legendre Tx

9) Canonical TX
   Treat $x, \dot{x}$ equally
   Poisson Bracket
   connection to QM

10) Hamilton - Jacobi Theory

11) Canonical P.T.

12) Field Theory

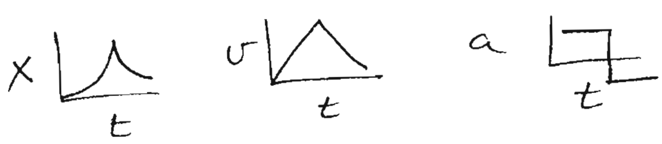
Traditional Approach

Goal: describe motion of obj

Input: I.C. and applied Forces

Need x, $\dot{x}$ and F

Why not $\ddot{x}$, $\dddot{x}$?



Not continous

Newton 2nd Law: $F = ma = m\ddot{x}$

$$\Rightarrow \quad x(t) = \int_0^t dt' \int_0^{t'} dt'' \; \frac{F(t'')}{m}$$

$$\vec{F} = m\vec{a} \equiv \frac{d\vec{P}}{dt}$$

General Observation:

$\vec{F} = 0 \Rightarrow P = \text{const}$ **Prove:**

Rot. Analogue:

$$\vec{\tau} = 0 \Rightarrow \vec{L} = \text{const}$$

Proof: $\tau = r \times F \qquad L = r \times P$

$$r \times \vec{F} = r \times \frac{dP}{dt}$$

$$\tau = \frac{dL}{dt}$$

Trivial Here. Generalize in Hamiltonian formulation

Cyclic coordinate $\Rightarrow$ const of motion

Work

$$W = F d$$

$$W_{12} = \int_1^2 \vec{F} \circ d\vec{s} = T_2 - T_1 \quad \text{for const. } m$$

work done by external force

$$F\,dx = m\ddot{x}\,dx = m\dot{v}\,dx = m\frac{dv}{dt}dx = m\left(\frac{dx}{dt}\right)dv = m v\,dv$$

$$\int \Rightarrow \tfrac{1}{2} m v^2$$

Special case

$$\vec{F} = -\vec{\nabla} V \sim -\frac{dV}{d\vec{s}}$$

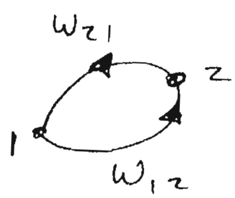
$$\vec{F} \circ d\vec{s} = -dV$$

$$\therefore \int_1^2 \vec{F} \circ d\vec{s} = -(V_2 - V_1) = V_1 - V_2 = W_{12} = T_2 - T_1$$

$$\therefore T_1 + V_1 = T_2 + V_2 \qquad E \text{ conservation}$$

Practical Result:

$$\oint F \cdot ds = \lim_{2\to1} \int_1^2 \vec{F} \cdot d\vec{s} = \lim (V_1 - V_2) = V_1 - V_1 = 0$$

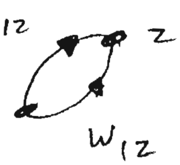


$$W_{12} + W_{21} = 0$$

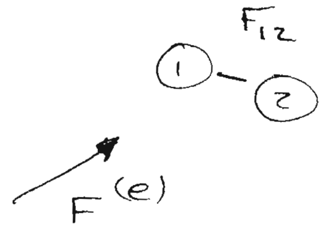
$$W_{12} = -W_{21}$$

$$\Rightarrow \quad W_{12} = +\widehat{W}_{12}$$

$$-W_{21} = \widehat{W}_{12}$$



System of Particles



$$\cancel{F_i = \dot{p} = F^{e} + \sum F_{ij}}$$

$$\sum_i \dot{P}_i = F = \sum_{i,j}{}' F_{ji} + \sum_i F_i^{(e)}$$

$$\downarrow$$

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i \equiv \frac{d^2}{dt^2} M \vec{R}_{CMS}$$

$$\dot{P} = m\dot{v} = m\ddot{x}$$

where $M = \sum m$ Tot M

$$R_{CMS} = \frac{\sum m_i \vec{r}_i}{M} \quad \text{CMS}$$

Newton's 3rd Law $\Rightarrow$ $F_{ij} + F_{ji} = 0$

internal Forces cancel

$$\vec{P}_{CM} = M \frac{d^2 \vec{R}_{CM}}{dt^2} = 0 + \sum_i F_i^{(e)} = F^{(e)}$$

Rot Analogue :

$$\frac{d\vec{L}_{CM}}{dt} = N^{(e)}$$

Constraints

holonomic	$F(r_i \dots, t) = 0$	holo: whole
non-holonomic	$F(r_i \dots, t) \geq 0$	
rheonomous	$F(r \dots, t)$	rheo: flowing
scleronomous	$F(r \dots, 0)$	sclero: hard

	h	n-h
r	$F(t) = 0$	$F(t) \geq 0$
S	$F(0) = 0$	$F(0) \geq 0$

$\boxed{\text{Examples}}$???

Simple Case

holonomic:

N particles

3N dimensions

K constraints

3N-K indep. coord. $\boxed{\text{Generalized coordinates}}$

Trans Form:

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{3N-K} \end{pmatrix} = \underbrace{\overset{T}{\begin{pmatrix} \quad \end{pmatrix}}}_{\leftarrow 3N \rightarrow} {\scriptstyle \updownarrow 3N-K} \begin{pmatrix} r_1 \\ \vdots \\ r_{3N} \end{pmatrix}$$

If linear $\Longrightarrow$

and scleronomous

Example: $x, y \Longrightarrow x - y = L$

$q_1 = x$ one ind.

Non-Holonomic

$X, Y,$

$X \geq 0$

can not eliminate variable

Also: Disk rolling on plane w/out slipping

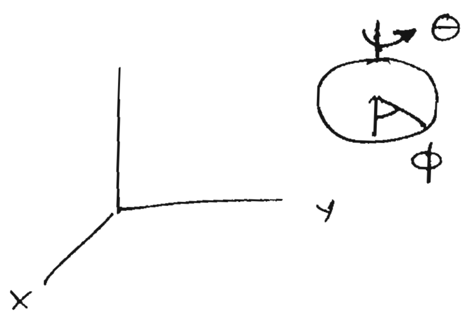
X, Y, ϕ, Θ

- X, Y c.m. of disk
- Θ orient.
- ϕ ang distance

For radius r:

$$r\dot{\phi} = \sqrt{\dot{x}^2 + \dot{y}^2} = v$$

can not eliminate variable.



Note, can go in loop

s.t. $\{X\ Y\ \Theta\}_0 = \{X\ Y\ \Theta\}_1$

but $\phi_0 \neq \phi_1$

Can not eliminate ϕ

D'Laemberts Princ

$$\dot{p} = F = F_{applied} + F_{const.} = F^{(a)} + F$$

Trick: Use work.

Why? Because F_{const} usually does no work

$$W = F \cdot \delta r = \vec{F}^{(a)} \cdot \delta \vec{r} + \underbrace{\vec{F} \cdot \delta \vec{r}}_{=0 \text{ usually}} = \dot{p} \cdot \delta r$$

Virtual displacement at Fixed time

$\therefore$

$$\boxed{\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \cdot \delta r_i = 0}$$

Change variables to q's

$$\sum_i \left[Q_i - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \right\} \right] \delta q_i = 0$$

Generalized Force

curvature of coordinates
(=0 for cartesian)

$T = \frac{1}{2} m v^2$

$\frac{\partial T}{\partial x} = 0$

$$Q_i = \sum_j \vec{F}_j^{(a)} \cdot \frac{\partial r_j}{\partial q_i}$$

Since above holds for any δq_i, and q_i are independent

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

For each i separately.

For the portion of F s.t. $F = -\nabla V$

$$\vec{F}_i = -\vec{\nabla} V_i$$

$$Q_i = \sum_j \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} = -\sum_j \vec{\nabla}_j V \cdot \frac{\partial \vec{r}_j}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Since $\frac{\partial V}{\partial \dot{q}} = 0$ V is independent of velocity

---

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \overset{\text{or}}{=} 0$$

For the remaining Forces only

$$L = T - V$$
