Newton's Law

$$F = ma = \mathring{p}$$

D'lambert's Principle

$$(\mathring{p} - F_A)\,\delta q = 0$$

Constraint forces do no work

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathring{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad L = T - V$$

Hamilton's Equations

Symmetric in q and p

$$\frac{\partial H}{\partial p} = +\mathring{q} \qquad \frac{\partial H}{\partial q} = -\mathring{p} \qquad H(q\,p) = \mathring{q}\,p - L(q\,\mathring{q})$$

Least Action Principle

Compact Statement

$$\delta S = 0 = \delta \int dt\, L(q\,\mathring{q})$$

$\boxed{\text{Newton's 2nd Law}}$ $F = ma = \dot{p}$

$$\dot{p} = F = F_{Applied} + F_{constraint}$$

Multiply by δr or δq to get to work

$$\dot{p}\,\delta q = F_A\,\delta q + \underbrace{F_c\,\delta q}_{\to\, = 0 \text{ in most cases}}$$

i.e., constraint Forces do no work

$\therefore$

$$(\dot{p} - F_A)\,\delta q \qquad \boxed{\text{D'lambert's Principle}}$$

Next, show above is same as $\boxed{\text{Lagrange Eqs.}}$

$$\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q}\right]\delta q - \left[\frac{d}{dt}\frac{\partial V}{\partial \dot{q}} - \frac{\partial V}{\partial q}\right]\delta q = 0$$

$T = \frac{1}{2} m \dot{q}^2$ recall $L = T - V$

$$\frac{\partial T}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial T}{\partial q} = 0 \qquad F = -\frac{\partial V}{\partial q} \qquad \frac{\partial V}{\partial \dot{q}} = 0$$

$$\to [m\ddot{q} - 0]\,\delta q - [0 + F]\,\delta q = 0$$

$$\therefore \quad (\dot{p} - F)\,\delta q = 0 \qquad \text{Q.E.D.}$$

Show Least Action is same as Lagrange Eqs.

$$\delta S = 0 = \delta \int dt\, L(q\,\dot{q})$$

$$= \int dt \left\{ \frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta\left(\frac{d}{dt} q\right) \right\}$$

Move using integration by parts, and pick up minus sign

$$= \int dt \left\{ \frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\,\delta q \right\} = 0$$

$$\therefore \quad \int dt\, (-\delta q) \left\{ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right\} = 0$$

Show Least Action $\Rightarrow$ Hamilton's Equations

$$\delta S = 0 = \delta \int dt\, L(q\dot{q}) = \delta \int dt \left[\dot{q} P - H(q P) \right]$$

$$\int dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right] = \int dt \left[\dot{q}\,\delta P + P\,\delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial P}\delta P \right]$$

③ ① ②

① $\Rightarrow \quad \dfrac{\partial L}{\partial \dot{q}} = P$

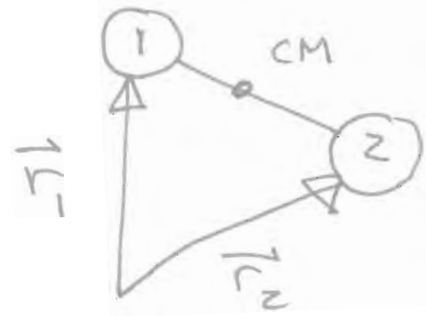
② $\Rightarrow \quad \dfrac{\partial H}{\partial q} = -\dfrac{\partial L}{\partial q} = -\dot{P}$

③ $\Rightarrow \quad \dfrac{\partial H}{\partial P} = +\dot{q}$

System of Particles Ch. 9

Center of Mass

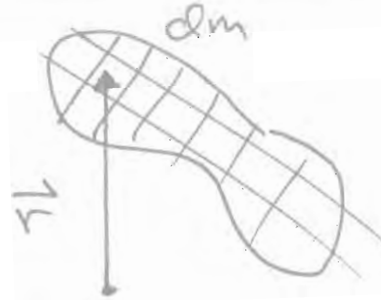
For discrete system:



$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

$$= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

For continuous system



$$\vec{R} = \frac{1}{M} \int dm \, \vec{r}$$

Consider 2-body system:

$$\dot{P}_1 = F_{12} + F_{External}$$

$$\dot{P}_2 = F_{21} + F_{External}$$

$\boxed{\text{Show internal Forces cancel}}$

$$\sum_i \dot{P}_i = \dot{P}_1 + \dot{P}_2 = 0 + F_{External}$$

$F_{12} + F_{21} = 0$ by Newton's 3rd La

For system of particles:

$$\sum_i \dot{P}_i = F_{External}$$

$$\sum_i \dot{P}_i = \sum_i m_i a_i = \frac{d^2}{dt^2}\left(\sum_i m_i \vec{r}_i\right) = \frac{d^2}{dt^2}\left(M \vec{R}_{CM}\right)$$

$$\therefore \quad \dot{P}_{CM} = M a_{CM} = F_{External} \qquad (9.13)$$

Motion of CM is determined by External Forces

Observation:

$$L = MR \times \dot{R} + \sum_\alpha r'_\alpha \times P'_\alpha = \underbrace{R \times P}_{} + \sum_\alpha r'_\alpha \times P'_\alpha$$

Angular momentum of CM about coordinate system origin

Angular Momentum of body about CM

Show internal Torques N cancel:

$$\dot{P}_{CM} = \sum_i \dot{P}_i = F_{external}$$

$$\dot{L} = \sum_i \dot{L}_i = \sum N = \underbrace{\sum N_{internal}}_{=0} + \sum N_{External}$$

Newton's 3rd Law

$$\therefore \quad \dot{L} = N_{External} \qquad (9.31)$$

Energy

$$T = \underbrace{\tfrac{1}{2} M V^2}_{\text{Energy of CM}} + \underbrace{\tfrac{1}{2} I \omega^2}_{\text{Energy due to motion of body about the CM}}$$

Big Picture

Decompose problems in 2 parts.

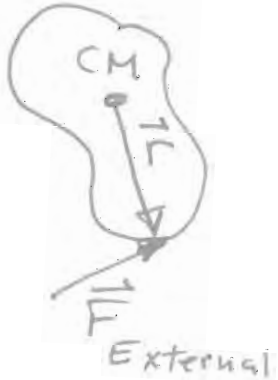
1. Determine motion of CM. (External Forces)

   $F_{External} \Rightarrow \dot{P}_{CM}$

   $N_{External} \Rightarrow \dot{L}_{CM}$

2. Determine motion about CM. (Internal Forces)

Example



$$\dot{P}_{CM} = \vec{F}_{External}$$

$$\dot{L}_{CM} = \vec{N}_{Ext} = \vec{r} \times \vec{F}_{EXT}$$

$\boxed{\text{$\vec{r}$ goes from origin to point of applied Force. Origin need not be CM}}$

$\boxed{\text{Elastic Collisions}}$ (E is conserved)

Work in 1-D First

If we know $\{m_1, m_2, v_1\ v_2\}$ Find v_1' and v_2'.

Need 2 equations: P conservation and E conservation

$\boxed{P\text{ cons.}}$

$$P_{cm} = P'_{cm}$$

$$M V_{cm} = M V'_{cm}$$ ← P cons. implies that V_{cm} is constant

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$m_1 (v_1 - v_1') = -m_2 (v_2 - v_2') \qquad \text{Eq. } 1$$

$\boxed{E\text{ cons.}}$

$$E = E'$$

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2$$

$$m_1 (v_1 - v_1')(v_1 + v_1') = -m_2 (v_2 - v_2')(v_2 + v_2') \qquad \text{Eq } 2$$

Divide Eq 2 / Eq 1 ⟹

$$v_1 + v_1' = v_2 + v_2'$$

$$(v_1 - v_2) = -(v_1' - v_2')$$

Velocity of approach equals velocity of recession

$\boxed{\text{Elastic Collisions}}$ 2-D

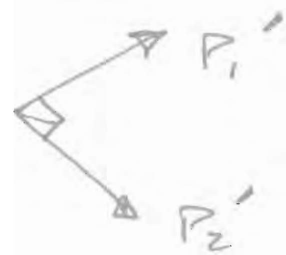
Assume $m_1 = m_2$, and $v_2 = 0$.

Show Angle between v_1' and $v_2' = 90^\circ$

(I.e., Show $P_1' \cdot P_2' = 0$)

$\boxed{\text{Before}}$

$\boxed{\text{After}}$



Use P and E conservation

$\boxed{\text{P. Cons}}$ $$\underbrace{P_1 + P_2}_{P_2 = 0} = P_1' + P_2'$$

$$P_1^2 = (P_1' + P_2')^2 = P_1'^2 + P_2'^2 + 2P_1' \cdot P_2' \quad \text{Eq } 1$$

$\boxed{\text{E cons}}$ $$T = \frac{1}{2} m v^2 = \frac{P^2}{2m}$$

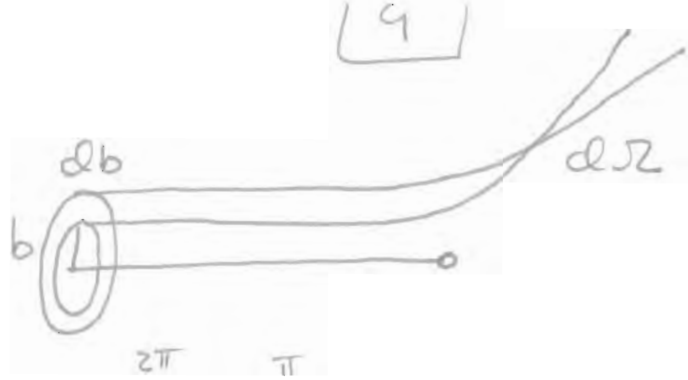
$$\frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} = \frac{P_1'^2}{2m_1} + \frac{P_2'^2}{2m_2}$$

If $m_1 = m_2$ and $P_2 = 0 \Rightarrow P_1^2 = P_1'^2 + P_2'^2$ (2)

Combine Eqs. 1 - Eq 2

$$\Rightarrow \quad 2P_1' \cdot P_2' = 0 \quad \text{I.e., } P_1' \perp P_2'$$

Cross Section



$$dN = I \frac{d\sigma}{d\Omega} d\Omega$$

$d\Omega$ is solid angle $\quad \int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta = 4\pi$

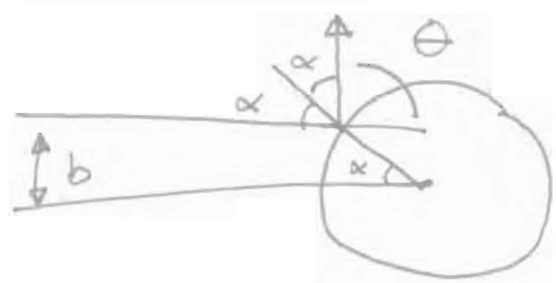
Observe: $\quad 2\pi b \, db = \frac{d\sigma}{d\Omega} d\Omega$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{2\pi b \, db}{d\Omega} = \frac{2\pi b \, db}{2\pi \sin\theta \, d\theta} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|$$

Assuming no ϕ-dependence

Example | Scattering From a hard sphere

Compute $\frac{db}{d\theta}$



① $2\alpha + \theta = \pi$

② $\sin\alpha = b/R$

① $\Rightarrow \quad \alpha = \frac{\pi - \theta}{2} \quad$ or $\quad \sin\alpha = \sin\left(\frac{\pi-\theta}{2}\right)$

② $\Rightarrow b = R \sin\alpha \equiv R\cos\left(\frac{\theta}{2}\right) \quad$ with some trig identities

$$db = -\frac{R}{2} \sin\left(\frac{\theta}{2}\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right| = \frac{R^2}{4}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{R^2}{4} = \pi R^2$$

Area of sphere

Chpt 2:

Hamiltons Principle : If $F = -\nabla V$ $\begin{cases} v \text{ dep.} \\ v \text{ ind} \end{cases}$

monogenic: where

Conservative : $F = -\nabla V$, time-ind.

$$\delta I = \delta \int_{t_1}^{t_2} L \, dt = 0$$

Extremum

Stationary

Not necess. Min or Max

— Why does nature pick classical path ?

Classical path is special

— Why <u>least</u> Action principle (misnomer)

Analogue of Fermat's principle

of geo. optics.

Light minimizes time

— Important for QFT

1-D case Free particle $V = 0$

$$\delta I \Rightarrow \delta \int T\, dt \Rightarrow \delta\left(\tfrac{1}{2} m v^2\right) = 0$$

$$\delta v = 0$$

(2.3) Derive Lagrange Eq. from Hamilton's Princ.

$$\delta I = \delta \int L(q, \dot{q}, t)\, dt$$

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \frac{\partial q}{\partial \alpha} d\alpha + \frac{\partial L}{\partial \dot{q}} \underbrace{\frac{\partial \dot{q}}{\partial \alpha}}_{\frac{d \partial q}{dt\, \partial \alpha}} d\alpha + 0 \right\} dt$$

Int. by parts

BC $\Rightarrow$ limits vanish

$$- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\left(\frac{\partial q}{\partial \alpha}\right)$$

$$\hookrightarrow \left(\frac{\partial L}{\partial \dot{q}}\right)\left(\frac{\partial q}{\partial \alpha}\right)\Bigg]_{t_1}^{t_0}$$

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \underbrace{\left[\left(\frac{\partial L}{\partial q}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]}_{\to\, 0 \quad \forall \alpha} \left(\frac{\partial q}{\partial \alpha}\right) d\alpha\, dt$$

arbitrary

Example : Brachistochrone Problem.

$$v = \frac{ds}{dt}$$

$$dt = \frac{ds}{v}$$

$$t = \int_1^2 \frac{ds}{v}$$

E cons $\Rightarrow$ $T + V = 0$

$$\tfrac{1}{2} m v^2 - m g y = 0$$

$$\Rightarrow v = \sqrt{2 g y}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\therefore \quad t = \int_1^2 \underbrace{\frac{\sqrt{1+\dot{y}}}{\sqrt{2gy}}}_{\to\, L(y,\dot{y})} dx$$

$$\frac{\partial L}{\partial y} = \frac{\sqrt{1+\dot{y}}}{\sqrt{2g}}\left(\frac{-y^{-3/2}}{2}\right) = \frac{-\sqrt{1+\dot{y}}}{4\sqrt{2gy}}\left(\frac{2}{y}\right)$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{(1+\dot{y})^{-1/2}}{2\sqrt{2gy}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{-1}{4\sqrt{2gy}\sqrt{1+\dot{y}}}\left[(1+\dot{y})\,\ddot{y} + \frac{\dot{y}}{y}\right]$$

Bead on Wire

$x = r\cos\omega t$

$y = r\sin\omega t$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ $\qquad V = 0 \qquad L = T - V = T$

$= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$

$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial r} = mr\omega^2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = 0$$

$$\boxed{m\ddot{r} - mr\omega^2 = 0}$$

$$\ddot{r} = r\omega^2$$

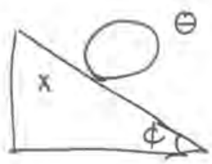
$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial r} = mr\dot{\theta}^2 \qquad \frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 = 0 \qquad mr^2\ddot{\theta} = 0}$$

$$\dot{\theta} = \omega$$

$\boxed{P.50}$



$r\theta = x$

$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \qquad V = -mg\,x \sin\phi$

$L = T - V$

$\boxed{\text{Method \#1}}$ use $r\theta = x \qquad \theta = x/r$

$\therefore \quad L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + mgx \sin\phi$

$\frac{\partial L}{\partial x} = mg \sin\phi \qquad \frac{\partial L}{\partial \dot{x}} = 2m\dot{x}$

$2m\ddot{x} - mg\sin\phi = 0$

$\ddot{x} = \frac{g \sin\phi}{2}$

$\boxed{\text{Method \#2}}$

$r\theta - x = 0$

$\underset{a_\theta}{(r)}\,d\theta + \underset{a_x}{(-1)}\,dx = 0$

$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + mgx \sin\phi$

$\frac{\partial L}{\partial x} = mg\sin\phi \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

① $m\ddot{x} - mg\sin\phi = (-1)\lambda$ ② $mr^2\ddot{\theta} - 0 = (r)\lambda$

③ $r\,d\theta - dx = 0$

③ $r\ddot{\theta} = \ddot{x}$

② $\Rightarrow$ $mr^2\ddot{\theta} = \lambda r$

$mr\ddot{\theta} = \lambda$

$m\ddot{x} = \lambda$

$\ddot{x} = \lambda/m$

① $\Rightarrow$ $\lambda = m\ddot{x}$ $\Rightarrow$ $2\lambda = mg\sin\phi$

$\lambda = mg\sin\phi/2 \quad = \frac{\ddot{x}}{m} = \frac{r}{m}\ddot{\theta}$

$\ddot{\ }$

Brachistochrone Problem

$$S = v t$$

$$t = \frac{S}{v}$$

$$t = \int_1^2 \frac{dS}{v}$$

$$2 g y = v^2$$

$$v = \sqrt{2 g y}$$

$$S = \sqrt{x^2 + y^2}$$

$$dS^2 = dx^2 + dy^2$$

$$\therefore \quad t = \int_1^2 \underbrace{\left[\frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2 g y}} \right]}_{L(y \dot{y})} dx$$

$$\delta I = 0 = \int dt \; L(q \dot{q})$$

$$= \int dx \; L(y, \dot{y}) \qquad \dot{y} = \frac{dy}{dx}$$