

Chpt 3

Kepler :

which $\Rightarrow F \propto 1/r^2$

- 1) Orbits are ellipses
- 2) Equal A wr Equal Time
- 3) $T^2 \propto r^3$

Newton

$$F = \frac{GMm}{r^2}$$

Choose Polar coords

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \quad V = ? \quad \text{Assume } V = V(r)$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

Note: θ is absent $\Rightarrow P_{\theta}$ is conserved:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \equiv \frac{d}{dt} P_{\theta} \Rightarrow P_{\theta} = \text{const}$$

$$P_{\theta} = \boxed{m r^2 \dot{\theta} = l} = \text{const}$$

For r eq

$$m \ddot{r} - m r \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0$$

$$\Rightarrow \boxed{m \ddot{r} - \frac{l^2}{m r^3} = - \frac{\partial V(r)}{\partial r}}$$

Use E -cons:

$$E = T + V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r)$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$$

$$\frac{dE}{dt} = \left[m \dot{r} \ddot{r} - \frac{l^2}{mr^3} + \frac{\partial V(r)}{\partial r} \right] \frac{dr}{dt} \stackrel{?}{=} 0$$

$$\Rightarrow E = \text{const}$$

∞

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$$

2nd Eq

Solve for \dot{r}

$$\dot{r}^2 = \frac{2}{m} \left(E - V(r) - \frac{l^2}{2mr^2} \right)$$

Solve for $\dot{\theta}$

$$\dot{\theta} = \frac{l}{mr^2}$$

4 const. of Int: E, l, r_0, θ_0

Int.
Consts.

$$\text{Show } \frac{dE}{dt} = 0$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2m} \frac{1}{r^2} + V$$

$$\text{Show } \frac{dE}{dt} = 0$$

$$\textcircled{1} \quad \frac{d}{dt} \left(\frac{1}{2} m \dot{r}^2 \right) = m \dot{r} \ddot{r}$$

$$\textcircled{2} \quad \frac{d}{dt} \left(\frac{l^2}{2m} r^{-2} \right) = - \frac{l^2}{m r^3} \dot{r}$$

$$\textcircled{3} \quad \frac{d}{dt} [V(r)] = \frac{dV}{dr} \frac{dr}{dt} = -F \dot{r}$$

$$\frac{dE}{dt} = \dot{r} \left[m \ddot{r} - \frac{l^2}{m r^3} - F \right]$$

$\underbrace{\hspace{10em}}$
Eq of motion for r

$$\ddot{r} \rightarrow 0 \Rightarrow \frac{dE}{dt} = 0$$

Sec 3-3 Effective 1-D Problem

Note: Eq. For r :

$$\dot{r}^2 = \frac{2}{m} \left(E - \left[V(r) + \frac{l^2}{2mr^2} \right] \right)$$

In analogue for a 1-D system

$$E = \frac{1}{2} m \dot{x}^2 + V(x)$$

$$\dot{x}^2 = \frac{2}{m} (E - V(x))$$

∞ This is a 1-D problem w/ the replacement:

$$V(r) \rightarrow V(r) + \frac{l^2}{2mr^2} = V(r) + \delta V(r)$$

Or, for the force: $F = -\nabla V$

$$\delta F = -\nabla(\delta V) = + \frac{l^2}{mr^3} = mr\dot{\theta}^2 = \frac{m\dot{\theta}^2}{r}$$

centrifugal force ↗

$$\infty F \rightarrow F + \delta F = F + \frac{m\dot{\theta}^2}{r}$$

we have chosen a non-inertial system

Likewise:

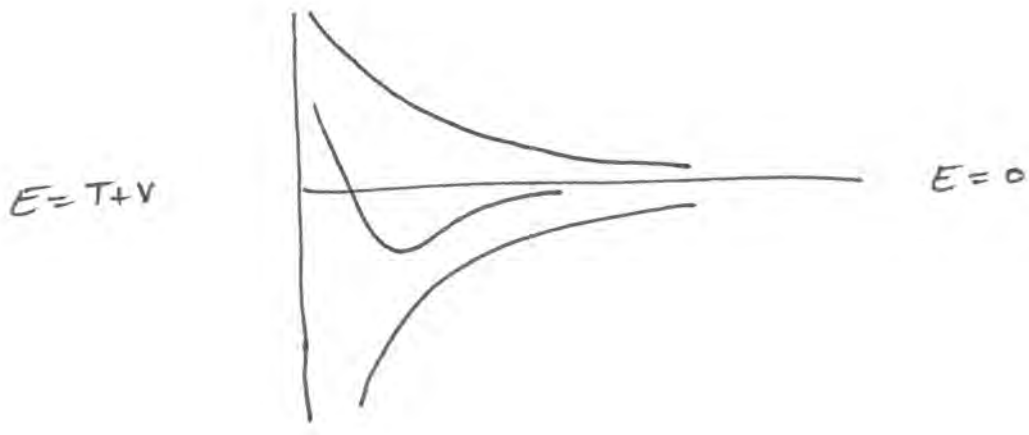
$$\hat{V}'(r)$$

$$E \rightarrow \frac{1}{2} m \dot{r}^2 + (V(r) + \delta V(r))$$

Consider Kepler force

$$F = -\frac{K}{r^2} \quad V = -\frac{K}{r}$$

$$\hat{V}' = -\frac{K}{r} + \frac{l^2}{2mr^2}$$



Viviani's Theorem:

$$\dot{p} = F$$

$$G = p \cdot r \quad (\text{No physical Interpretation})$$

$$\frac{dG}{dt} = \frac{d}{dt} (p \cdot r) = \dot{p} \cdot r + p \cdot \dot{r} = F \cdot r + 2T$$

$\hookrightarrow F \cdot r \quad \hookrightarrow 2T = mv^2$

Time Average

$$\frac{1}{\tau} \int_0^{\tau} dt \frac{dG}{dt} = \frac{1}{\tau} [G(\tau) - G(0)] = \overline{F \cdot r} + \overline{2T}$$

For periodic motion of period $\tau \rightarrow = 0$

$$\Rightarrow \overline{T} = -\frac{1}{2} \overline{F \cdot r} \quad (\text{Not True w/out Aver})$$

For Cons. Forces: $F = -\nabla V = -\frac{\partial V}{\partial r}$

~~$\Rightarrow \overline{T} = -\frac{1}{2} \overline{F \cdot r}$~~

$$\text{For } V(r) = a r^{n+1} \Rightarrow \overline{T} = \frac{n+1}{2} \overline{V}$$

$$V = \frac{k}{r} \quad \Rightarrow \quad n = -2 \quad \overline{T} = -\frac{1}{2} \overline{V}$$

Sec 3-4 :

Given

$$\dot{r}^2 = \frac{2}{m} T' \quad \dot{\theta} = \frac{l}{mr^2}$$

$$T' = E - \left(V + \frac{l^2}{2mr^2} \right) \quad \text{EFF KE}$$

Eliminate t variable :

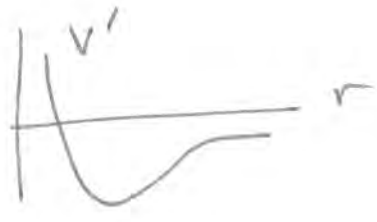
$$\frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\infty \quad \frac{d}{dt} = \frac{l}{mr^2} \frac{d}{d\theta}$$

$$\infty \quad \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \left(\frac{l}{mr^2} \right) = \sqrt{\frac{2}{m} T'}$$

$$\frac{dr}{d\theta} = \sqrt{\frac{2mr^4}{l^2} T'} = \frac{mr^2}{l} \sqrt{\frac{2}{m} T'}$$

Closed Orbits :



$$V' = V + \frac{l^2}{2mr^2}$$

$$V'(r) = V'(r_0) + \left. \frac{\partial V'}{\partial r} \right|_{r_0} (r-r_0) + \frac{1}{2} \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r_0} (r-r_0)^2$$

Assume $V = -\frac{K}{r^n}$ $\frac{\partial V}{\partial r} = \frac{+nK}{r^{n+1}}$ $\frac{\partial^2 V}{\partial r^2} = \frac{-n(n+1)K}{r^{n+2}}$

For Min, $\left. \frac{\partial V'}{\partial r} \right|_{r_0} = 0 \Rightarrow \frac{nK}{r^{n+1}} = \frac{l^2}{mr^3}$

For bounded orbit. $\left. \frac{\partial^2 V'}{\partial r^2} \right|_{r_0} > 0 \Rightarrow$

$$-\frac{n(n+1)K}{r^{n+2}} + \frac{3l^2}{mr^4} > 0$$

$\hookrightarrow -\frac{(n+1)l^2}{mr^4} + 3 \frac{l^2}{mr^4} > 0$ From $\left. \frac{\partial V'}{\partial r} \right|_{r_0} = 0$

$$-(n+1) + 3 > 0$$

$$\boxed{2 > n}$$

Consider V' harmonic potential

$$V'(r) = V'(r_0) + \frac{1}{2} C (\Delta r)^2$$

$$C = \left. \frac{\partial^2 V'}{\partial r^2} \right|_{r_0} = [-(n+1)+3] \frac{\hbar^2}{m r^4} = (2-n) \frac{\hbar^2}{m r^4}$$

SHO $T = \frac{1}{2} m \dot{r}^2$ $V = \frac{1}{2} \kappa r^2$

$$L = T - V$$

$$E_L \Rightarrow m \ddot{r} + \kappa r = 0$$

$$r(t) = r_{\text{AMP}} e^{i\omega t} \quad \omega^2 = \frac{\kappa}{m}$$

Shift by r_0

$$r(t) = r_0 + r_{\text{AMP}} e^{i\omega t}$$

$$= r_0 + r_{\text{AMP}} \cos \omega t$$

Need relation of t and θ

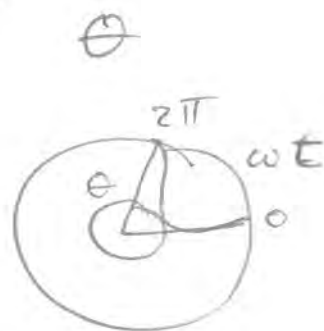
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$$\dot{\theta} = \frac{l}{mr^2} \Rightarrow \theta = t \left(\frac{l}{mr^2} \right)$$

$$\omega t = \sqrt{\frac{k}{m}} \left(\frac{mr^2}{l} \right) \theta$$

$$= \sqrt{(2-n) \frac{l^2}{mr^4} \cdot \frac{1}{m} \frac{m^2 r^4}{l^2}}$$

$$\omega t = \sqrt{(2-n)} \theta$$



$$r(t) = r_0 + r_{AMP} \cos \sqrt{2-n} \theta$$

Orbit is periodic if $\sqrt{2-n} = \text{rat} = \frac{p}{q}$

Examples

$$V = \frac{-k}{r} \quad n=1 \quad \sqrt{2-n} = 1$$

$$V = \frac{-k}{r^{-2/9}} \quad n = \frac{2}{9} \quad \sqrt{2 - \frac{2}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

$$V = \frac{1}{2} k r^2 \quad n = -2 \quad \sqrt{2 - (-2)} = 4$$

Kepler Problem

$$\text{For } v = \frac{-k}{r}$$

$$\frac{1}{r} = C(1 + e \cos \theta)$$

$$C = \frac{mK}{L^2}$$

$$r = \frac{1}{C(1 + e \cos \theta)}$$

$$e = \sqrt{1 + \frac{2EL^2}{mK^2}}$$

Classify orbits in terms of e

$e < 1$ bounded

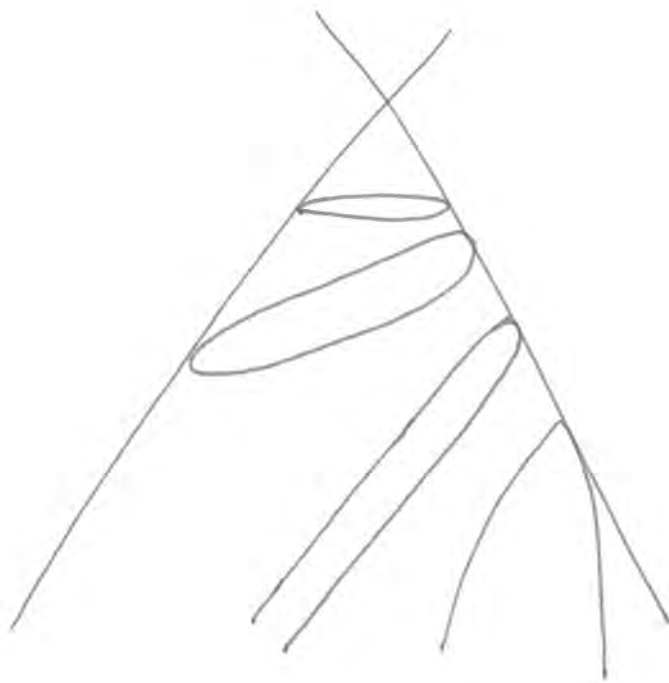
$e > 1$ unbounded $r \rightarrow \infty$

$e > 1$ $E > 0$ hyperbola

$e = 1$ $E = 0$ parabola

$0 < e < 1$ $E < 0$ ellipse

$e = 0$ $E = -\frac{mK^2}{2L^2}$ circle



For $V = -K/r^n$ at min r_0 $\frac{nK}{r^{n+1}} = \frac{l^2}{mr^3}$

$\Rightarrow V(r_0) = -\frac{K}{r_0^n} = -\frac{1}{n} \frac{l^2}{mr^2}$

For $n=1$ $V(r_0) = -\frac{l^2}{mr^2} \equiv -\frac{K}{r_0}$

For circular orbit $E = -\frac{mK^2}{2l^2} = -\frac{K}{2r_0} = \frac{V}{2}$

so $T = -\frac{V}{2}$ aka Virial Theorem

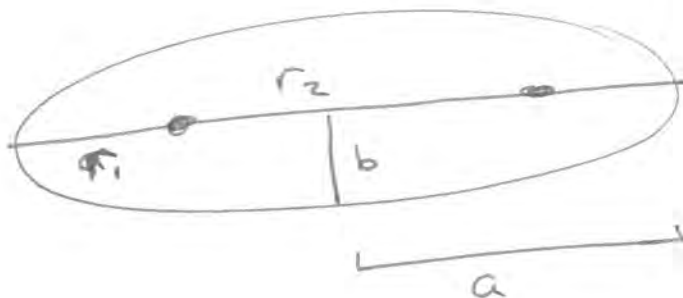
Turning points:

$E = T + \frac{K}{r} + \frac{l^2}{2mr^2}$

For $T=0 \rightarrow r^2 \frac{E}{E} + \frac{K}{E} r - \left(\frac{l^2}{2m}\right) \frac{1}{E} = 0$

r_{\pm} = roots of \rightarrow

$a = \frac{r_1 + r_2}{2} = \frac{+K}{2(-E)}$



$r_2 = a(1+e)$

$r_1 = a(1-e)$

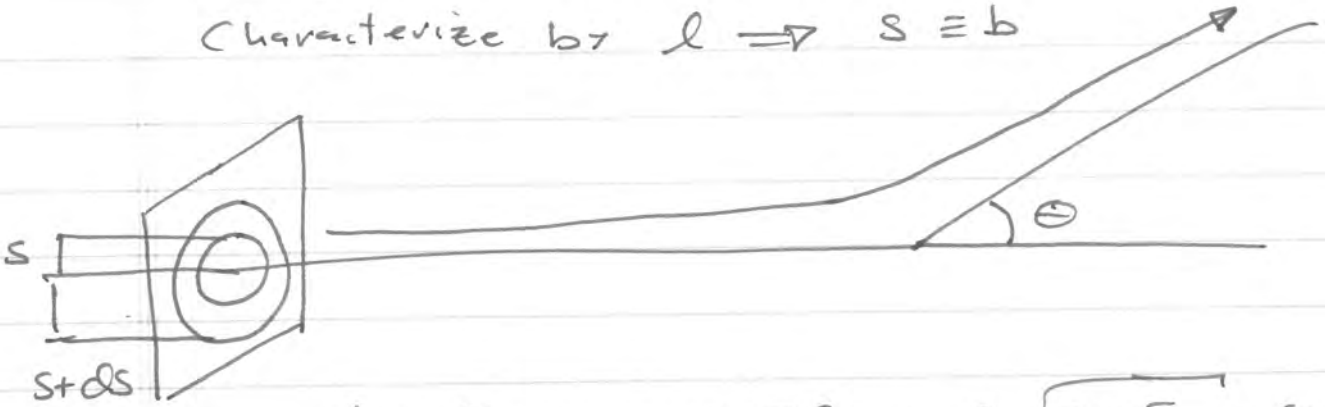
$r_1 + r_2 = 2a$

Scattering in Central Force Field

$$N \propto d\sigma \cdot I \cdot d\Omega$$

$$d\Omega = 2\pi \sin\theta d\theta$$

Characterize by $l \Rightarrow s \equiv b$



$$l = P \times r = m v s = s \sqrt{2mE} = \text{const}$$

$$\text{Number thro area} = 2\pi s ds I$$

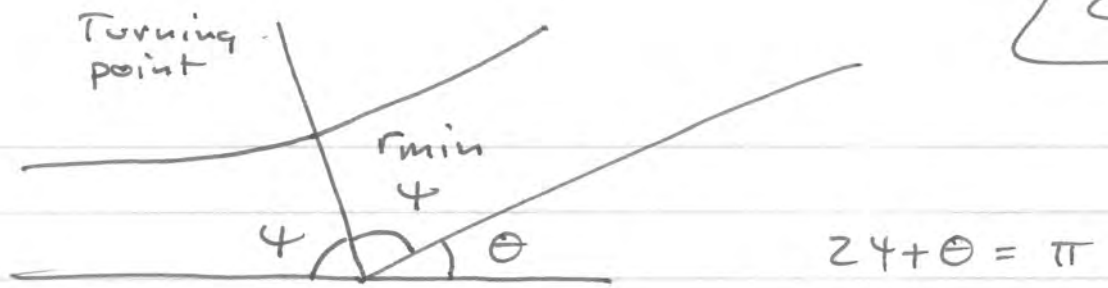
$$\text{Number thro } d\Omega = d\Omega d\sigma I$$

Particle conservation

$$\Rightarrow 2\pi s ds I = 2\pi \sin\theta d\theta d\sigma I$$

$$s ds = \sin\theta d\theta d\sigma$$

$$d\theta = \frac{s}{\sin\theta} \left| \frac{ds}{d\sigma} \right|$$



$$\frac{dr}{dt} = F(r, t) \Rightarrow \frac{dr}{d\theta} = F(r, \theta)$$

$$\Rightarrow d\theta = \frac{dr}{F(r, \theta)}$$

Orbit Equation
 r_{min}, ∞
 are Turning Pts

$$\psi = \int_{r_{min}}^{\infty} \frac{dr}{F(r)} = \int_{r_m}^{\infty} \frac{dr}{r^2 \sqrt{\frac{2ME}{l^2} - \frac{2MV}{l^2} - \frac{1}{r^2}}}$$

$$\theta = \pi - 2\psi$$

$$\psi = \frac{\pi - \theta}{2}$$

For repulsive Coulomb

$$F = + \frac{zz'e^2}{r^2}$$

$$\rightarrow K = -zz'e^2$$

$$e = \sqrt{1 + \frac{2El^2}{mK^2}} \Rightarrow \sqrt{1 + \frac{2El^2}{m(zz'e^2)^2}}$$

$$e = \sqrt{1 + \left(\frac{2Es}{zz'e^2} \right)^2}$$

Just use Std. Kepler Solution

$$\infty \quad C = \frac{m z z' e^2}{r^2}$$

General
Kepler Solution

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$$\frac{1}{r} = C \frac{(\epsilon \cos \phi - 1)}{\phi}$$

$$\boxed{\psi \equiv \phi}$$

$$r \rightarrow \infty \quad \epsilon \cos \phi = 1 \Rightarrow \cos \phi = \frac{1}{\epsilon}$$

$$r \rightarrow r_{\min} \quad \cos \phi = 0$$

$$\infty \quad \cos \psi = \frac{1}{\epsilon} = \cos\left(\frac{\pi - \theta}{2}\right) = \sin\left(\frac{\theta}{2}\right)$$

~~Relate~~ Relate θ to s

$$\epsilon^2 - 1 = \frac{1 - s^2}{s^2} = \cot^2 \frac{\theta}{2}$$



$$\left(\frac{2 \epsilon s}{z z' e^2} \right) = \cot \frac{\theta}{2} = \frac{\cos \theta/2}{\sin \theta/2}$$

$$\left| \frac{ds}{d\theta} \right| = \left(\frac{z z' e^2}{2 \epsilon} \right) \frac{1}{\sin^2 \theta/2} \cdot \frac{1}{2}$$

∞

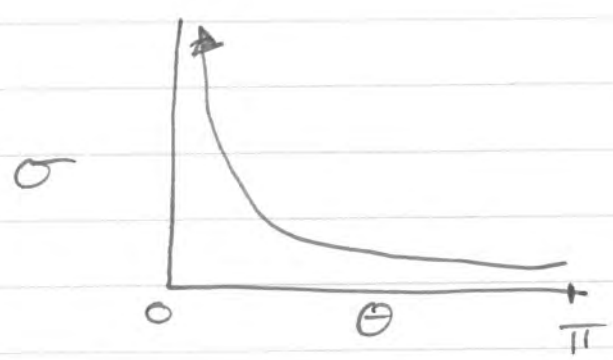
$$d\sigma = \frac{s}{\sin \theta} \left| \frac{ds}{d\theta} \right| = \left(\frac{z z' e^2}{2 \epsilon} \right)^2 \frac{\cos \theta/2}{\sin^3 \theta/2} \cdot \frac{1}{\sin \theta} \cdot \frac{1}{2}$$

$$= \frac{1}{4} \left(\frac{z z' e^2}{2 \epsilon} \right)^2 \frac{1}{\sin^4 \theta/2}$$

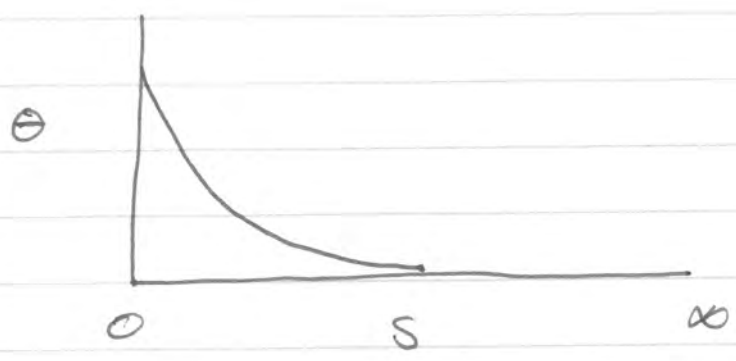
$$\sigma = \int \rho \, d\Omega = 2\pi \int_0^\pi \rho \sin \theta \, d\theta$$

$$= 2\pi C \int_0^\pi \frac{\sin \theta}{\sin^4 \theta/2} \, d\theta =$$

$$= 2\pi \left[\frac{1}{4} \left(\frac{z z' e^z}{z E} \right)^2 \right] \left[\frac{|-z|}{\sin^2(\theta/2)} \right]_0^\pi$$



$$S \left(\frac{z E}{z z' e^z} \right) = \cot \theta/2$$



$$S \rightarrow \infty \quad \theta \rightarrow \pi$$

$$\text{Show } \frac{dE}{dt} = 0$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2m} \frac{1}{r^2} + V$$

$$\text{Show } \frac{dE}{dt} = 0$$

$$\textcircled{1} \quad \frac{d}{dt} \left(\frac{1}{2} m \dot{r}^2 \right) = m \dot{r} \ddot{r}$$

$$\textcircled{2} \quad \frac{d}{dt} \left(\frac{l^2}{2m} r^{-2} \right) = - \frac{l^2}{m r^3} \dot{r}$$

$$\textcircled{3} \quad \frac{d}{dt} [V(r)] = \frac{dV}{dr} \frac{dr}{dt} = -F \dot{r}$$

$$\frac{dE}{dt} = \dot{r} \left[m \ddot{r} - \frac{l^2}{m r^3} - F \right]$$

$\underbrace{\hspace{10em}}$
Eq of motion for r

$$\ddot{r} \rightarrow 0 \quad \Rightarrow \quad \frac{dE}{dt} = 0$$