Dimensional Regularization

meets

Freshman E&M

References:
Fredrick Olness, arXiv:0812.3578
M. Hans, Am.J.Phys. 51 (8) August (1983). p.694
C. Kaufman, Am.J.Phys. 37 (5), May (1969) p.560
B. Delamotte, Am.J.Phys. 72 (2) February (2004) p.170

Fred Olness
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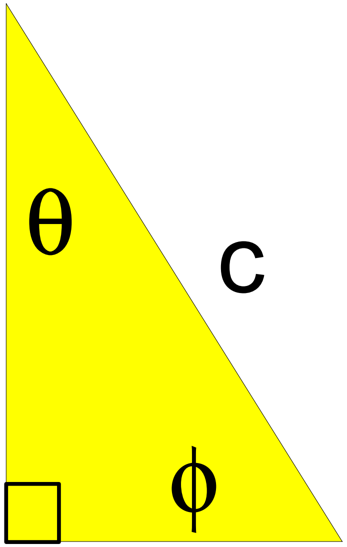
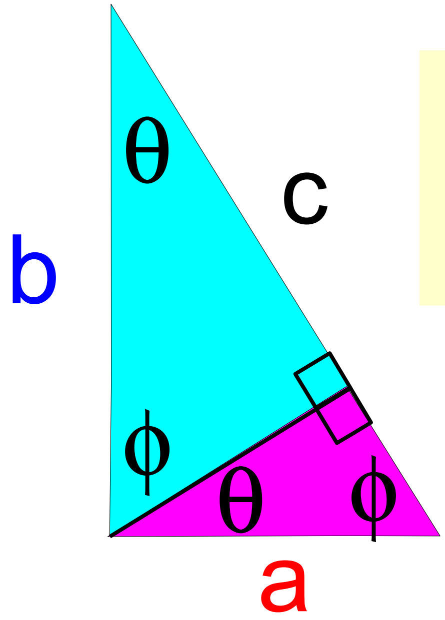
Pythagorean Theorem

GOAL:
Pythagorean Theorem

METHOD:
Dimensional Analysis



$$A_c = c^2 f(\theta, \phi)$$

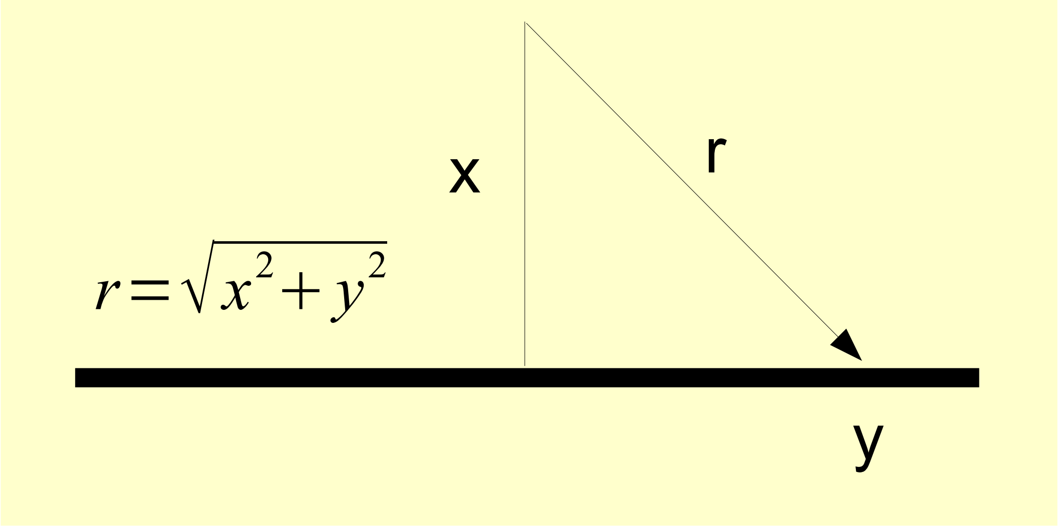


$$A_a + A_b =$$
$$a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

$$A_a + A_b = A_c$$

$$a^2 + b^2 = c^2$$

Infinite Line of Charge

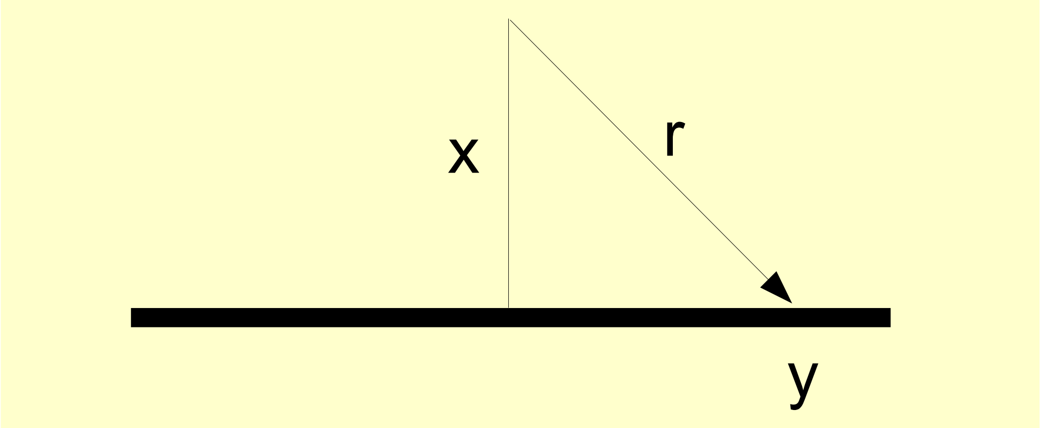


$$dV = \frac{1}{4\pi\epsilon_0} \; \frac{dQ}{r} \qquad\qquad \lambda = Q / y$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \; \frac{1}{\sqrt{x^2 + y^2}} = \infty$$

Note: ∞ can be very useful

Scale Invariance



$$V(kx) =$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dy \; \frac{1}{\sqrt{(kx)^2 + y^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} d\left(\frac{y}{k}\right) \frac{1}{\sqrt{x^2 + (y/k)^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dz \; \frac{1}{\sqrt{x^2 + z^2}}$$
$$= V(x)$$

$$V(kx) = V(x)$$

Note: $\infty + c = \infty$

$\infty - \infty = c$

Cutoff Method

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{+L} dy \frac{1}{\sqrt{x^2 + y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+L + \sqrt{L^2 + x^2}}{-L + \sqrt{L^2 + x^2}}\right]$$

- V(x) depends on artificial regulator L
- We cannot remove the regulator L

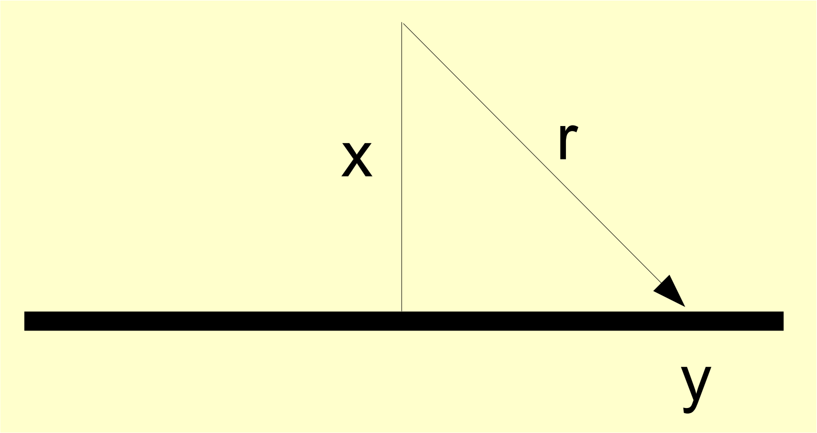
All physical quantities are independent of the regulator:

Electric Field $$E(x) = \frac{-dV}{dx} = \frac{\lambda}{2\pi\epsilon_0 \, x} \frac{L}{\sqrt{L^2 + x^2}} \rightarrow \frac{\lambda}{2\pi\epsilon_0 \, x}$$

Energy $$\delta V = V(x_1) - V(x_2) \underset{L \rightarrow \infty}{\rightarrow} \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale L
AND we have a broken symmetry: translation invariance

Broken Translational Symmetry



Shift: $y \to y' = y - c$

y=[+L+c, -L+c]

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L+c}^{+L+c} dy \; \frac{1}{\sqrt{x^2 + y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \log\left[\frac{+(L+c) + \sqrt{(L+c)^2 + x^2}}{-(L-c) + \sqrt{(L-c)^2 + x^2}}\right]$$

V(r) depends on y coordinate!!!

Dimensional Regularization

Compute in n-dimensions

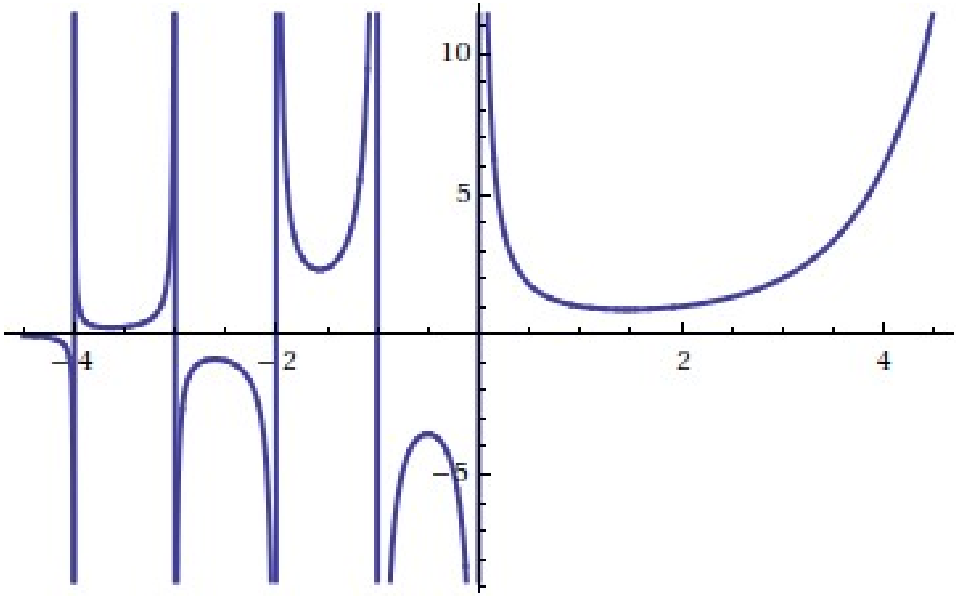
$$dy \to d^n y = \frac{d\,\Omega_n}{2}\ y^{n-1}\, dy$$

$$\Omega_n = \int d\,\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

$$\Omega_{1,2,3,4} = \{2, 2\pi, 4\pi, 2\pi^2\}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^{+\infty} d\,\Omega_n\ \frac{y^{n-1}}{\mu^{n-1}}\ \frac{dy}{\sqrt{x^2+y^2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma[\epsilon]}{\pi^{\epsilon}} \right)$$



Dimensional Regularization

All physical quantities are independent of the regulators:

Electric Field
$$E(x)=\frac{-dV}{dx}=\frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\mu^{2\epsilon}\Gamma[\epsilon]}{\pi^{\epsilon}x^{1+2\epsilon}}\right]\underset{\epsilon\to\infty}{\to}\frac{\lambda}{2\pi\epsilon_0}\frac{1}{x}$$

Energy
$$\delta V=V(x_1)-V(x_2)\underset{\epsilon\to\infty}{\to}\frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{x_2^2}{x_1^2}\right]$$

Problem solved at the expense of an extra scale μ and regulator ε

Translation invariance is preserved!!!

Dimensional Regularization respects symmetries

Renormalization

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\left[\frac{\mu^2}{x^2}\right]\right]$$

Original

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\left[\frac{\mu^2}{x^2}\right]\right]$$

MS

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon}+\ln\left[\frac{e^{-\gamma_E}}{\pi}\right]+\left[\frac{\mu^2}{x^2}\right]\right]$$

MS-Bar

$$V_{\overline{MS}}(x_1)-V_{\overline{MS}}(x_2)=\delta V=V_{MS}(x_1)-V_{MS}(x_2)$$

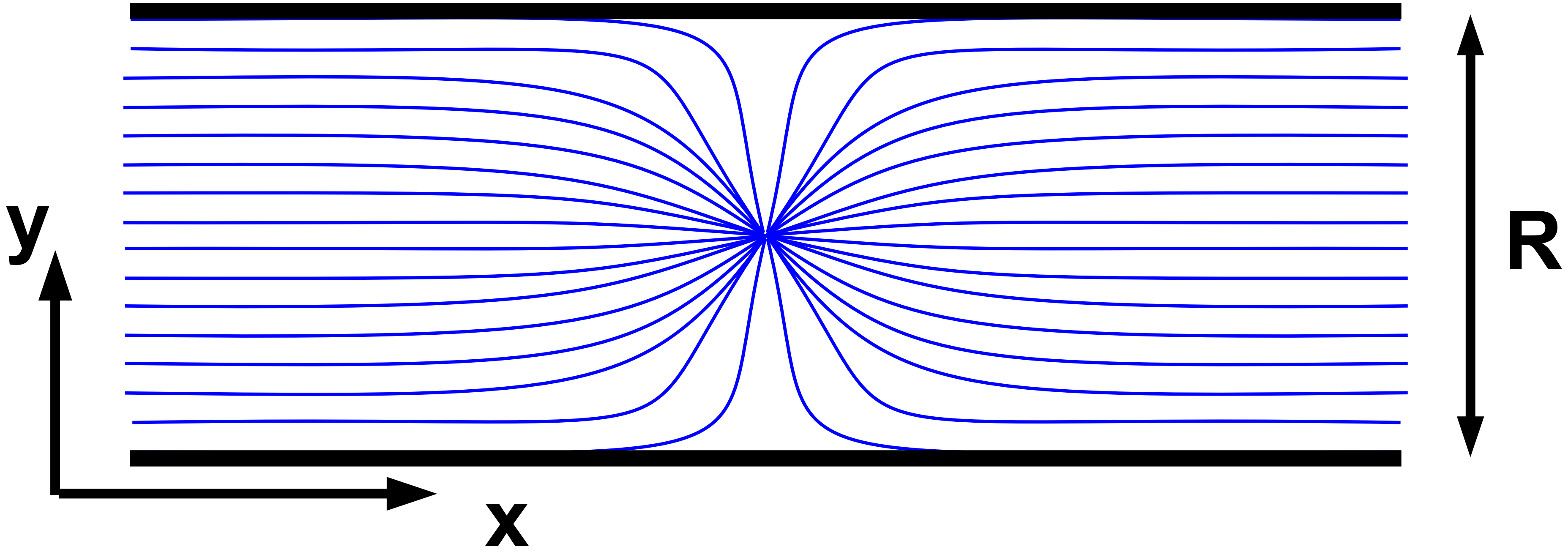
$$V_{\overline{MS}}(x_1)-V_{MS}(x_2)\neq\delta V\neq V_{MS}(x_1)-V_{\overline{MS}}(x_2)$$

Connection to QFT

$$V \rightarrow \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \left[\frac{\mu^2}{x^2}\right] \right]$$

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\mu^2}{Q^2}\right) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \rightarrow \frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{4\pi}\right] + \left[\frac{\mu^2}{Q^2}\right]$$

Dimensional Transmutation



$$V(r) \sim \frac{1}{r^{D-2}} \qquad E(r) \sim \frac{1}{r^{D-1}}$$

Renormalization Group Equation

$$\sigma = f \otimes \omega$$

$$\frac{d\sigma}{d\mu} = 0 = \frac{df}{d\mu}\,\omega + f\,\frac{d\omega}{d\mu}$$

$$\frac{1}{\tilde{f}}\,\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = -\frac{1}{\tilde{\omega}}\,\frac{d\tilde{\omega}}{d\ln[\mu]}$$

$$\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma\,\tilde{f} \qquad\qquad \frac{df}{d\ln[\mu]} = P \otimes f$$

$$\tilde{f} \sim \mu^{-\gamma}$$

Recap

Regulator provides unique definition of V, f, ω

Cutoff regulator L:
simple, but does NOT respect symmetries

Dimensional regulator ε:
respects symmetries: translation, Lorentz, Gauge invariance
introduces new scale μ

All physical quantities (E, dV, σ) are independent of the regulator
Renormalization group equation: $d\sigma/d\mu=0$

We can define renormalized quantities (V,f,ω)
Renormalized (V,f,ω) are scheme dependent and arbitrary
Physical quantities (E,dV, σ) are unique and scheme independent
if we apply the scheme consistently