Recap: Parton Model, Factorization, Evolution

How does $f$ change with scale $\mu$??

$\mu$ dependence must balance

DGLAP Evolution Equation

$\frac{1}{\bar{f}} \frac{d\bar{f}}{d\ln[\mu]} = -\gamma$

Electron

Proton

Sample NLO contributions to DIS

DIS at NLO
DIS NLO Kinematics

**Mandelstam Variables \{s, t, u\}**

\[ s = (k_1 + k_2)^2 = (k_3 + k_4)^2 \]
\[ t = (k_1 - k_3)^2 = (k_2 - k_4)^2 \]
\[ u = (k_1 - k_4)^2 = (k_2 - k_3)^2 \]

\[ s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \]

Exercise

\{s, t, u\} are partonic

Homework Part 2

**Problem #2**: Consider the reaction: \( pp \to pp \) (12 → 34) with CMS scattering angle \( \theta \). The CMS energy is \( \sqrt{s} = 2 \, TeV \).

a) Compute the boost from the CMS frame to the rest frame of \#2 (lab frame).

b) Compute the energy of \#1 in the lab frame.

c) Compute the scattering angle \( \theta_{ab} \) as a function of the CMS \( \theta \) and invariants.

Hint: by using invariants you can keep it simple. I.e., don’t do it the way Goldstein does.

The power of invariants

---

**Exercise**

1) Let’s work out the general 2→2 kinematics for general masses.

a) Start with the incoming particles.

Show that these can be written in the general form:

\[ p_1 = (E_1, 0, 0, +p) \]
\[ p_2 = (E_2, 0, 0, -p) \]

… with the following definitions:

\[ E_{1,2} = \frac{\hat{s} \pm m_1^2 \pm m_2^2}{2\sqrt{s}} \]
\[ p = \frac{\Delta(\hat{s}, m_1^2, m_2^2)}{2\sqrt{s}} \]

\[ \Delta(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)} \]

Note that \( \Delta(a, b, c) \) is symmetric with respect to its arguments, and involves only the invariants of the initial state: \( s, m_1^2, m_2^2 \).

b) Next, compute the general form for the final state particles, \( p_3 \) and \( p_4 \). Do this by first aligning \( p_3 \) and \( p_4 \) along the z-axis (as \( p_1 \) and \( p_2 \) are), and then rotate about the y-axis by angle \( \theta \).
Matrix element: NLO DIS

\[
|\mathcal{M}|^2 = \frac{s}{-t} + \frac{-t}{s} + \frac{2\mu Q^2}{st} = \frac{2(1-x)}{(1-z)} + \frac{2(1-z)}{(1-x)} + \frac{2x(1+z)}{(1-x)(1-z)}
\]

For the real 2→2 graphs

Singular at \( z=1 \)
\( z \to 1, \quad \cos \theta \to 1 \)
\( \theta \to 0, \quad t \to 0 \)

Collinear Singularity

Separate infinity, and subtract

Singular at \( x=1 \)
\( x \to 1, \quad s \to 0 \)

Soft Singularity

Separate infinity, cancel with virtual graphs

The Plan

1) Separate \( \infty \) at \( z=1 \)
2) Subtract … (should be part of PDF)

Need to regulate \( \infty \)

Collinear Divergences

Plan

Method

Choices

1) Dimensional Regularization
2) Quark Mass
3) \( \theta \) Cut

Soft Singularities

Plan

Method

Choices

1) Separate \( \infty \) at \( x=1 \)
2) Cancel between Real and Virtual graphs

Need to regulate \( \infty \)

1) Dimensional Regularization
2) Gluon Mass
3) ...
We'll use a simple example to illustrate the key points:

Dimensional Regularization meets Freshman E&M

\[ V(x) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} dy \frac{1}{\sqrt{(kx)^2 + y^2}} \]

\[ V(x) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} dz \frac{1}{\sqrt{x^2 + z^2}} \]

Note: \( \infty + c = \infty \)

\[ V(kx) = V(x) \]

Naively implies: \( V(kx) - V(x) = 0 \)

\[ dV = \frac{1}{4\pi\varepsilon_0} \frac{dQ}{r} \]

\[ \lambda = Q/y \]

\[ V = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} dy \frac{1}{\sqrt{x^2 + y^2}} = \infty \]

Note: \( \infty \) can be very useful

\[ V(x) \text{ depends on artificial regulator } L \]

We cannot remove the regulator \( L \)

All physical quantities are independent of the regulator:

Electric Field

\[ E(x) = -\frac{dV}{dx} = \frac{\lambda}{2\pi\varepsilon_0} x \frac{L}{\sqrt{L^2 + x^2}} \to \frac{\lambda}{2\pi\varepsilon_0} x \]

Energy

\[ \delta V = V(x_1) - V(x_2) \to \frac{\lambda}{4\pi\varepsilon_0} \log \left[ \frac{x_2^2}{x_1^2} \right] \]

Problem solved at the expense of an extra scale \( L \)

**AND** we have a broken symmetry: translation invariance.
Broken Translational Symmetry

Shift: \( y \rightarrow y' = y - c \)

\( y = [+L+c, -L+c] \)

\[
V = \frac{\lambda}{4\pi \varepsilon_0} \int_{-L-c}^{L+c} dy \frac{1}{\sqrt{x^2 + y^2}}
\]

\[
V = \frac{\lambda}{4\pi \varepsilon_0} \log \left[ \frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}} \right]
\]

V(r) depends on “y” coordinate!!!

In QFT, gauge symmetries are important. E.g., Ward identities

Why do we need an extra scale \( \mu \) ???

\[
v = \sqrt{x^2 + y^2}
\]

\[
dV = \frac{1}{4\pi \varepsilon_0} \frac{dQ}{r} \quad \lambda = Q/y
\]

\[
V = \frac{\lambda}{4\pi \varepsilon_0} f(x)
\]

Dimensional Regularization

Compute in n-dimensions

\[
dy \rightarrow d^n y = \frac{d \Omega_n}{2} y^{n-1} dy
\]

\[
\Omega_n = \int d \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}
\]

\[
\Omega_{1,2,3,4} = [2, 2\pi, 4\pi, 2\pi^2]
\]

\[
V = \frac{\lambda}{4\pi \varepsilon_0} \int_{0}^{\infty} d \Omega_n \frac{y^{n-1}}{\mu^{n-1}} \frac{dy}{\sqrt{x^2 + y^2}}
\]

\[
n = 1 - 2\epsilon
\]

New scale \( \mu \)

All physical quantities are independent of the regulators:

Electric Field

\[
E(x) = \frac{-dV}{dx} = \frac{\lambda}{4\pi \varepsilon_0} \left[ \frac{2\epsilon \mu^{2\epsilon} \Gamma(1+\epsilon)}{\pi^{\frac{1}{2}}} \right] \quad \epsilon \rightarrow 0 \quad \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{x}
\]

Energy

\[
\delta V = V(x_1) - V(x_2) \quad \epsilon \rightarrow 0 \quad \frac{\lambda}{4\pi \varepsilon_0} \log \left[ \frac{x_2^2}{x_1^2} \right]
\]

Problem solved at the expense of an extra scale \( \mu \) AND regulator \( \epsilon \)

Translation invariance is preserved!!!

Dimensional Regularization respects symmetries
Renormalization

\[ V \rightarrow \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{1}{\epsilon} + \ln \left( \frac{e^{-\gamma_e}}{\pi} \right) + \ln \left( \frac{\mu^2}{x^2} \right) \right] \]

Original

\[ V \rightarrow \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{1}{\epsilon} + \ln \left( \frac{e^{-\gamma_e}}{\pi} \right) + \ln \left( \frac{\mu^2}{x^2} \right) \right] \]

MS

\[ V \rightarrow \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{1}{\epsilon} + \ln \left( \frac{e^{-\gamma_e}}{\pi} \right) + \ln \left( \frac{\mu^2}{x^2} \right) \right] \]

MS-Bar

Physical quantities are independent of renormalization scheme!

\[ V_{\text{MS}}(x_1) - V_{\text{MS}}(x_2) \equiv \delta V = V_{\text{MS}}(x_1) - V_{\text{MS}}(x_2) \]

But only if performed consistently:

\[ V_{\text{MS}}(x_1) - V_{\text{MS}}(x_2) \neq \delta V \neq V_{\text{MS}}(x_1) - V_{\text{MS}}(x_2) \]

Recap

Regulator provides unique definition of \( V, f, \omega \)

- Cutoff regulator \( L \):
  - simple, but does NOT respect symmetries

- Dimensional regulator \( \epsilon \):
  - respects symmetries: translation, Lorentz, Gauge invariance
  - introduces new scale \( \mu \)

All physical quantities (\( E, dV, \sigma \)) are independent of the regulator

AND the new scale \( \mu \)

Renormalization group equation: \( d\sigma/d\mu=0 \)

We can define renormalized quantities (\( V, f, \omega \))

- Renormalized (\( V, f, \omega \)) are scheme dependent and arbitrary
- Physical quantities (\( E, dV, \sigma \)) are unique and scheme independent
  if we apply the scheme consistently

Apply Dimensional Regularization to QFT
Soft Singularities

D-Dimensional Phase Space

\[ d\sigma = \frac{1}{2s} |M|^2 \, d\Gamma \]

\[ d\Gamma_i = \frac{d^D k_i}{(2\pi)^D} (2\pi) \delta^D(k_i^2) \quad \text{1-particle} \]

\[ d\Gamma = d\Gamma_3 \, d\Gamma_4 \, (2\pi)^D \delta^D(k_1 + k_2 - k_3 - k_4) \quad \text{Final state} \]

\[ d\Gamma = \frac{1}{16\pi} \left( \frac{s}{16\pi} \right)^{-\epsilon} \frac{1}{\Gamma[1 - \epsilon]} \, d\mu \]

\[ g \rightarrow g \mu^\epsilon \quad \text{Enter, } \mu \text{ scale} \]

All the pieces

\[ d\Gamma = \frac{1}{16\pi} \left( \frac{16\pi \mu^2}{Q^2} \right)^{\epsilon} \left( \frac{1}{\Gamma[1 - \epsilon]} \right) \frac{x^\epsilon}{(1 - x)^{\epsilon}} \, d\mu \]

Homework: Part 1

#1) Show:

\[ \frac{d^3 p}{(2\pi)^3 2E} = \frac{d^4 p}{(2\pi)^4} (2\pi)^4 \delta^4(p^2 - m^2) \]

This relation is often useful as the RHS is manifestly Lorentz invariant

Note, we are working with massless partons, and \( \theta \) is in the partonic CMS frame

#2) Show that the 2-body phase space can be expressed as:

\[ d\Gamma = \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) = \frac{d \cos(\theta)}{16\pi} \]

Note, we are working with massless partons, and \( \theta \) is in the partonic CMS frame

Soft Singularities

\[ \frac{x^\epsilon}{(1 - x)^\epsilon} \]

From phase space

\[ \frac{1}{(1 - x)} \]

Soft Singularity

\[ \frac{1}{(1 - x)^+} - \frac{1}{\epsilon} \delta(1 - x) \]

Finite remainder

To be canceled by virtual diagram

This only makes sense under the integral

\[ \frac{f(x)}{(1 - x)^+} - \frac{f(x) - f(1)}{(1 - x)} \]

\[ \int_0^1 dx \, f(x) \frac{x^\epsilon}{(1 - x)^{1+\epsilon}} = \int_0^1 dx \, \frac{f(x) - f(1)}{(1 - x)} - \frac{1}{\epsilon} \int_0^1 dx \, \delta(1 - x) f(x) \]
Collinear Singularities

How do we know what goes in $\omega$ and PDFs ???

Compute NLO Subtractions for a partonic target

\[
\int_{-1}^{1} dz (1 - z^2)^{-\epsilon} |M|^2 \simeq -\frac{1}{\epsilon} \frac{(1 + x^2)}{(1 - x)} + \frac{1 - 4x + 4(1 + x^2) \ln 2}{2(1 - x)}
\]

This looks like part of the PDF

This is finite for $x=[-1,1]$

---

Key Points

1) Subtract

2) This is defined by the scheme

3) Need to match schemes of $\omega$ and PDF
   … MS, MS-Bar, DIS, …

4) Note we have regulator $\epsilon$ and extra scale $\mu$
Basic Factorization Formula

**At Zeroth Order:**
\[ \sigma^0 = f^0 \otimes \omega^0 + O(\Lambda^2/Q^2) \]

Use: \( f^0 = \delta \) for a parton target.
Therefore:
\[ \sigma^0 = f^0 \otimes \omega^0 = \delta \otimes \omega^0 = \omega^0 \]

**Warning:** This trivial result leads to many misconceptions at higher orders

---

**At First Order:**
\[ \sigma^1 = f^1 \otimes \omega^0 + f^0 \otimes \omega^1 \]
\[ \sigma^1 = f^1 \otimes \sigma^0 + \omega^1 \]

We used: \( f^0 = \delta \) for a parton target.
Therefore:
\[ \omega^1 = \sigma^1 - f^1 \otimes \sigma^0 \]

\[ f^1 \sim \frac{\alpha_s}{2\pi} \text{P}(1) \]

**At Second Order** *(NNLO)*:
\[ \sigma^2 = f^2 \otimes \omega^0 \otimes d^0 + \ldots \]
\[ + f^1 \otimes \omega^1 \otimes d^0 + \ldots \]

Therefore:
\[ \omega^2 = \ldots \]

**Combined Result:**
\[ \omega^0 + \omega^1 = \sigma^0 + \sigma^1 - f^1 \otimes \sigma^0 \]

**TOT**
**LO**
**NLO**
**SUB**
Subtraction

**TOT = LO + NLO - SUB**

---

**HOMEWORK PROBLEM:** NNLO Wilson Coefficients
Use the Basic Factorization Formula

\[ \sigma = f \otimes \omega + O(\Lambda^2/Q^2) \]

**At Second Order** *(NNLO)*:

Compute \( \omega^2 \) at second order.
Make a diagrammatic representation of each term.
Do we get different answers with different schemes???

NO !!!
NLO Theoretical Calculations:
   Essential for accurate comparison with experiments
We encounter singularities:
   Soft singularities: cancel between real and virtual diagrams
   Collinear singularities: “absorb” into PDF
Regularization and Renormalization:
   Regularize & Renormalize intermediate quantities
   Physical results independent of regulators (e.g., L, or $\mu$ and $\epsilon$)
   Renormalization introduces scheme dependence (MS-bar, DIS)
Factorization works:
   Hard cross section $\bar{\sigma}$ or $\omega$ is not the same as $\sigma$
   Scheme dependence cancels out (if performed consistently)