Convolutions:

Additive Version

$$f \circledast g$$

$$= \int dx \int dy \; f(x) \; g(y) \; \delta[a-(x+y)]$$

$$= \int dx \; f(x) \; g(a-x) \qquad \text{w/} \quad y = a-x$$

$$= \int dy \; f(a-y) \; g(y) \qquad \text{w/} \quad x = a-y$$

Convolution is symmetric!!!

Multiplicative Version

$$f \otimes g =$$

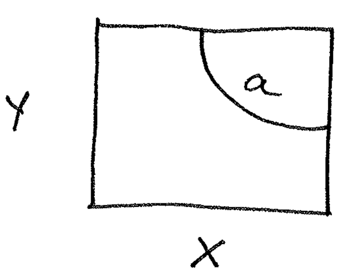
$$= \int dx \int dy \; f(x) \; g(y) \; \delta[a - xy]$$

$$= \int \frac{dx}{x} \; f(x) \; g\left(\frac{a}{x}\right) \qquad \text{w/} \quad y = \frac{a}{x}$$

$$= \int \frac{dy}{y} \; f\left(\frac{a}{y}\right) \; g(y) \qquad \text{w/} \quad x = \frac{a}{y}$$

Special Case ~~f, g defined $\in [0,1]$~~ $a, x, y \in [0,1]$

$y = \frac{a}{x}$ then $\Rightarrow$ $x \in [a, 1]$

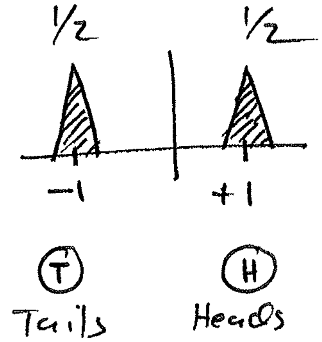


$\therefore$

$$f \otimes g = \int_a^1 \frac{dx}{x} \; f(x) \; g\left(\frac{a}{x}\right)$$

Example: Coin Flip:

$$f(x) = g(y) = \frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right] =$$



Double Coin Flip: $f \circledast g$

$$f \circledast g = \int dx \int dy \; f(x)\, g(y)\, \delta[a - (x+y)]$$

$$= \int dx \; f(x)\, g(a-x)$$

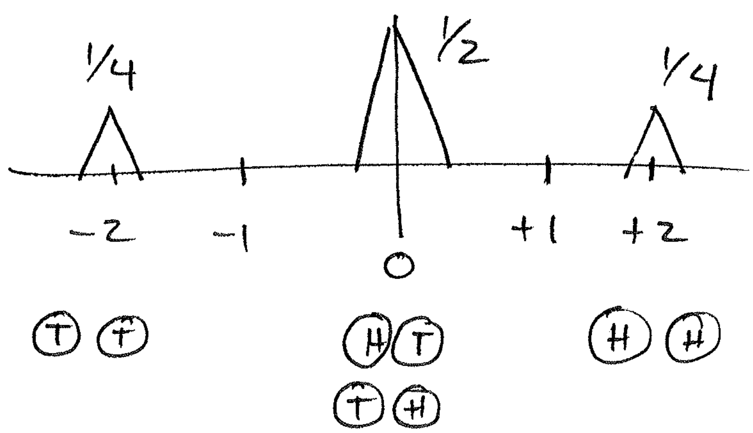
$$= \int dx \; \frac{1}{2}\left[\delta(x-1) + \delta(x+1)\right] \cdot \frac{1}{2}\left[\delta(a-x-1) + \delta(a-x+1)\right]$$

$$= \frac{1}{4}\int dx \; \Big\{ \delta(x-1)\left[\delta(a-x-1) + \delta(a-x+1)\right] + \delta(x+1)\left[\delta(a-x-1) + \delta(a-x+1)\right] \Big\}$$

$$= \frac{1}{4}\Big\{ \delta(a-2) + \delta(a) + \delta(a) + \delta(a+2) \Big\}$$

$$= \frac{1}{4}\Big\{ \delta(a-2) + 2\,\delta(a) + \delta(a+2) \Big\}$$

$=$



Example 2:

Compute average value of $F(x) = \frac{1}{2}[\delta(x-1)+\delta(x+1)]$

$$\int dx \; x \; F(x) = \int dx \; x \cdot \frac{1}{2}[\delta(x-1)+\delta(x+1)]$$

$$= \cancel{\int dx} \; \frac{1}{2}[(+1) + (-1)] = 0$$

Compute average squared $\overline{x^2}$:

$$\int dx \; x^2 \; F(x) = \int dx \; x^2 \cdot \frac{1}{2}[\delta(x-1)+\delta(x+1)]$$

$$\cancel{\langle x^2 \rangle} = \quad = \cancel{\int dx} \cdot \frac{1}{2}[(+1)^2 + (-1)^2] = 1$$

Compute $\overline{x^2}$ of $F \otimes F = \frac{1}{4}\{\delta(x-2) + 2\delta(x) + \delta(x+2)\}$

$$\int dx \; x^2 \; (F \otimes F) = \int dx \; x^2 \cdot \frac{1}{4}\{\delta(x-2)+2\delta(x)+\delta(x+2)\}$$

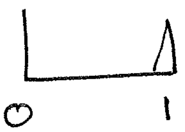
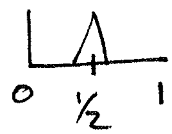
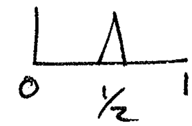
$$= \quad = \cancel{\int dx} \; \frac{1}{4}\{(+2)^2 + 2(0)^2 + (-2)^2\}$$

$$= \frac{8}{4} = 2$$

$$\langle \cancel{(F \otimes F)}^2 \rangle = \langle x^2 \rangle = 2$$

Example

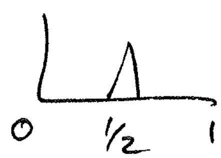
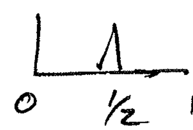
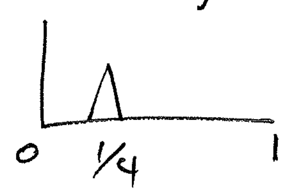
$$f \otimes g = \delta(1-x) \otimes \delta(y-\tfrac{1}{2}) = \delta(a-\tfrac{1}{2})$$

$$= \int \frac{dx}{x} f(x)\, g\!\left(\frac{a}{x}\right) = \int \frac{dx}{x} \delta(1-x)\, \delta\!\left(\frac{a}{x}-\frac{1}{2}\right)$$

$$= \delta(a-\tfrac{1}{2})$$

$$g \otimes g = \delta(x-\tfrac{1}{2}) \otimes \delta(y-\tfrac{1}{2}) = \delta(a-\tfrac{1}{4})$$

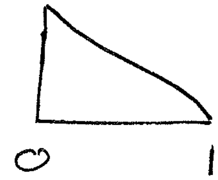
$$= \int \frac{dx}{x} f(x)\, g\!\left(\frac{a}{x}\right) = \int \frac{dx}{x} \delta(x-\tfrac{1}{2})\, \delta\!\left(\frac{a}{x}-\frac{1}{2}\right)$$

$$= 2\, \delta(2a-\tfrac{1}{2}) = \delta(a-\tfrac{1}{4})$$

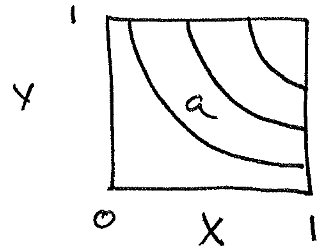
What do you expect for $g \otimes g \otimes g$

Example:

$F \otimes F = ?$ $\qquad f(x) = (1-x) =$



$$F \otimes F = \int_0^1 dx \int_0^1 dy \; F(x) \, F(y) \, \delta(a - xy)$$



$y = \frac{a}{x}$

Limits on x: $\in [a, 1]$

$\therefore$

$$F \otimes F = \int_a^1 \frac{dx}{x} \, F(x) \, F\left(\frac{a}{x}\right) =$$

$$= \int_a^1 \frac{dx}{x} (1-x) \left(1 - \frac{a}{x}\right) =$$