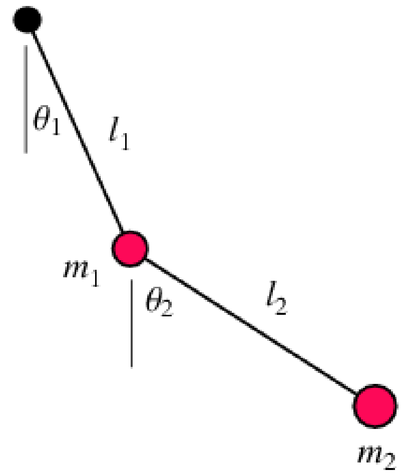
Double Pendulum



A double pendulum consists of one pendulum attached to another. Double pendula are an example of a simple physical system which can exhibit chaotic behavior. Consider a double bob pendulum with masses m_1 and m_2 attached by rigid massless wires of lengths l_1 and l_2. Further, let the angles the two wires make with the vertical be denoted θ_1 and θ_2, as illustrated above. Finally, let gravity be given by *g*. Then the positions of the bobs are given by

$$x_1 = l_1 \sin\theta_1 \quad (1)$$

$$y_1 = -l_1 \cos\theta_1 \quad (2)$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad (3)$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2. \quad (4)$$

The potential energy of the system is then given by

$$V = m_1 g y_1 + m_2 g y_2 \quad (5)$$

$$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2, \quad (6)$$

and the kinetic energy by

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \quad (7)$$

$$= \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]. \quad (8)$$

The Lagrangian is then

$$L \equiv T - V$$

$$= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+(m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2. \quad (9)$$

Therefore, for θ_1,

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \quad (10)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \quad (11)$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g (m_1 + m_2) \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \quad (12)$$

so the Euler-Lagrange differential equation for θ_1 becomes

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + l_1 g (m_1 + m_2) \sin\theta_1 = 0. \quad (13)$$

Dividing through by l_1, this simplifies to

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g (m_1 + m_2) \sin\theta_1 = 0. \quad (14)$$

Similarly, for θ_2,

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \quad (15)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \quad (16)$$

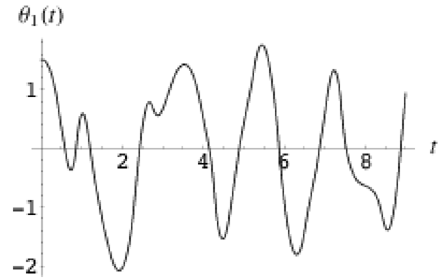
$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2, \quad (17)$$

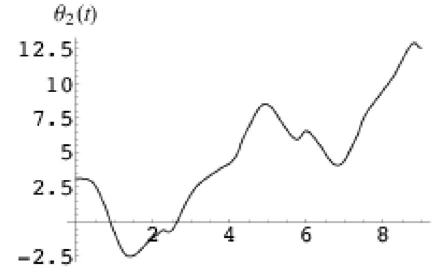
so the Euler-Lagrange differential equation for θ_2 becomes

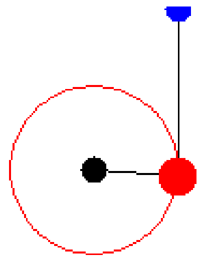
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + l_2 m_2 g \sin\theta_2 = 0. \quad (18)$$

Dividing through by l_2, this simplifies to

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \quad (19)$$







The coupled second-order ordinary differential equations (14) and (19) can be solved numerically for $\theta_1(t)$ and $\theta_2(t)$, as illustrated above for one particular choice of parameters and initial conditions. Plotting the resulting solutions quickly reveals the complicated motion.

The equations of motion can also be written in the Hamiltonian formalism. Computing the generalized momenta gives

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1-\theta_2) \quad (20)$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2). \quad (21)$$

The Hamiltonian is then given by

$$H = \theta_i p_i - L = \tfrac{1}{2}(m_1+m_2)l_1^2\theta_1^2 + \tfrac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1-\theta_2)$$

$$-(m_1+m_2)g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2. \quad (22)$$

Solving (20) and (21) for $\dot{\theta}_1$ and $\dot{\theta}_2$ and plugging back in to (22) and simplifying gives

$$H = \frac{l_2^2 m_2 p_{\theta_1}^2 + l_1^2(m_1+m_2)p_{\theta_2}^2 - 2 m_2 l_1 l_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1-\theta_2)}{2 l_1^2 l_2^2 m_2 [m_1 + \sin^2(\theta_1-\theta_2) m_2]}$$

$$- m_2 g l_2 \cos\theta_2 - (m_1+m_2) g l_1 \cos\theta_1. \quad (23)$$

This leads to the Hamilton's equations

$$\dot{\theta}_1 = \frac{\partial H}{\partial p_{\theta_1}} = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} \cos(\theta_1-\theta_2)}{l_1^2 l_2 [m_1 + m_2 \sin^2(\theta_1-\theta_2)]} \quad (24)$$

(25)

$$\dot{\theta}_2 = \frac{\partial H}{\partial p_{\theta_2}} = \frac{l_1(m_1+m_2)p_{\theta_2} - l_2 m_2 p_{\theta_1}\cos(\theta_1-\theta_2)}{l_1 l_2^2 m_2[m_1+m_2\sin^2(\theta_1-\theta_2)]}$$

$$\dot{p}_{\theta_1} = -\frac{\partial H}{\partial \theta_1} = -(m_1+m_2)gl_1\sin\theta_1 - C_1 + C_2 \quad (26)$$

$$\dot{p}_{\theta_2} = -\frac{\partial H}{\partial \theta_2} = -m_2 g l_2 \sin\theta_2 + C_1 - C_2, \quad (27)$$

where

$$C_1 \equiv \frac{p_{\theta_1} p_{\theta_2}\sin(\theta_1-\theta_2)}{l_1 l_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \quad (28)$$

and

$$C_2 \equiv \frac{l_2^2 m_2 p_1^2 + l_1^2(m_1+m_2)p_2^2 - l_1 l_2 m_2 p_1 p_2 \cos(\theta_1-\theta_2)}{2 l_1^2 l_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]^2}\sin[2(\theta_1-\theta_2)]. \quad (29)$$

SEE ALSO: Coupled Pendula, Hamiltonian, Lagrangian, Pendulum

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