Ch 4

Rigid Bodies: 6 - DOF $\left\{ \begin{array}{l} 3 \text{ CMS} \\ 3 \text{ Orient.} \end{array} \right.$

$$x' = R\,x$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1'\cdot x_1 & x_1'\cdot x_2 & x_1'\cdot x_3 \\ x_2'\cdot x_1 & x_2'\cdot x_2 & x_2'\cdot x_3 \\ x_3'\cdot x_1 & x_3'\cdot x_2 & x_3'\cdot x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 21 & r_{22} & r_{23} \\ 31 & 32 & r_{33} \end{pmatrix}$$

6 conditions

$$\alpha_\ell \alpha_m + \beta_\ell \beta_m + \gamma_\ell \gamma_m = \delta_{\ell m}$$

$$(x_1'\cdot x_1)(x_2'\cdot x_1) + (x_1'\cdot x_2)(x_2'\cdot x_2) + (x_1'\cdot x_3)(x_2'\cdot x_3) = 0$$

$$\hat{x}_{\substack{1\\2}}' = (x_{\substack{1\\2}}'\cdot x_{\substack{1\\1}})\,x_{\substack{1\\1}} + (x_{\substack{1\\2}}'\cdot x_{\substack{2\\2}})\,x_{\substack{2\\2}} + (x_{\substack{1\\2}}'\cdot x_{\substack{3\\3}})\,x_{\substack{3\\3}}$$

$$\Rightarrow x_1 = (x_1\cdot x_1')\,x_1' + (x_1\cdot x_2')\,x_2' + (x_1\cdot x_3')\,x_3'$$

$$x_2 = \quad 2\;1\;1 \quad 2\;2\;2 \quad 2\;3\;3$$

$$(x_1\cdot x_2) = (11')(21') + (12')(22') + (13')(23') = 0$$

$$(x_1'\cdot x_2') = (11')(12') + (21')(22') + (31')(32') = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & & \\ & & \end{pmatrix}$$

$$x'_i = a_{ij}\, x_j \quad \Rightarrow \quad \mathbf{X}' = A\, \mathbf{X}$$

$$\cancel{x'_i \cdot x'_k = \delta_{ik} \Rightarrow}$$

$$\cancel{a_{ij}\, x_j \; a_{kl}\, x_l = \delta_{ik}}$$

Want $\| X \|$ invariant

$$\Rightarrow \quad X \cdot X = X' \cdot X'$$

$$X_i X_i = X'_i X'_i = a_{ij} X_j \; a_{ik} X_k$$

$$\Rightarrow \quad a_{ij}\, a_{ik} = \delta_{jk}$$

$$A^T A = \mathbb{1}$$

$$X' = A X$$

$$X \cdot X = X^T X = X' \cdot X' = (X')^T X'$$

$$= (AX)^T (AX)$$

$$= X^T A^T A X$$

$$\Rightarrow \quad A^T A = \mathbb{1}$$

Orthogonal

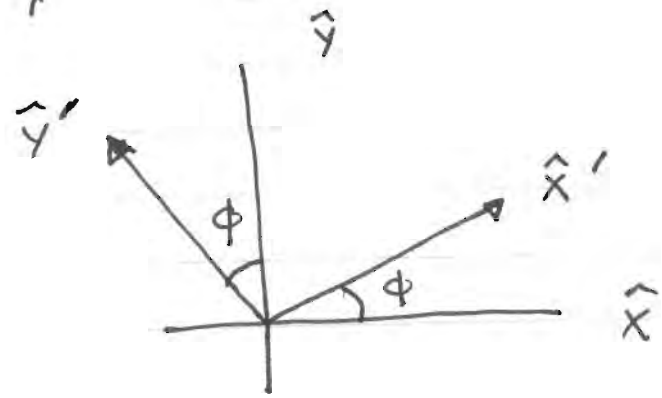
2-D Example

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

verify $A^T A = \mathbb{1}$

$$x' = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}$$
$$y' = -\sin\, \hat{x} + \cos\phi\, \hat{y}$$



Matrix Operations

1) $\exists \; \mathbb{1}$ s.t. $A \mathbb{1} = \mathbb{1} A = A$

2) $\exists \; A^{-1}$ s.t. $A^{-1} A = \mathbb{1}$

3) Assoc. $(AB)C = A(BC)$

Note: Not Comm $[A, B] \neq 0$

4) Dist: $A(B+C) = AB + AC$

$A^T A = \mathbb{1} \Rightarrow A^T = A^{-1}$ Orthogonal

$A^\dagger A = \mathbb{1} \Rightarrow A^\dagger = A^{*T} = A^{-1}$ Unitary

Adjoint

IF $A = A^T$ Self-adjoint
Hermitian

Eigenvalues of Herm. are real

Similarity Tx:

$$|X_2\rangle = H|X_1\rangle$$

Tx coords w/ R

$$R|X_2\rangle = RH|X_1\rangle = (RHR^{-1})(R|X_1\rangle)$$

$$\Rightarrow |X_2'\rangle = H'|X_1'\rangle$$

Det:

$$\|A \cdot B\| = \|A\| \, \|B\|$$

$$\Rightarrow \|A^T A\| = \|A^T\| \, \|A\| = \|A\|^2 = 1$$

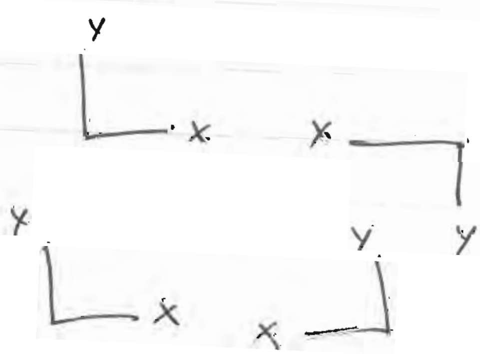
$$\Rightarrow \|A\| = \pm 1 \quad \Rightarrow \prod_{i=1}^{n} \lambda_i = \pm 1$$

Consider:

$$A = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$ Parity TX $\Rightarrow$ improper

2-D $\quad A = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

$A = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$



Euler Angles :

3 parameters : $\Rightarrow$ 3 rotations

$3 \times 2 \times 2$ choices $= 12$ conventions

1) Rotate about z
2) x'
3) z''

Caley – Klein Parameters:

2×2 representation: (complex)

8 - parameters:

$$x'^2 = x^2 = \langle x | x \rangle = \langle x' | Q^\dagger Q | x' \rangle \Rightarrow Q^\dagger Q = \mathbb{1}$$

$$\Rightarrow \quad Q = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

4 pars. $+$ $\mathrm{Det}\, Q = 1 = |\alpha|^2 + |\beta|^2$

How does $Q|x\rangle$?

$$|x\rangle = P = \begin{pmatrix} z & x - iy \\ x+iy & -z \end{pmatrix} = \begin{pmatrix} z & x_- \\ x_+ & -z \end{pmatrix} = \vec{\sigma} \cdot \vec{x}$$

$$|x'\rangle = A|x\rangle \Rightarrow P' = Q P Q^\dagger$$

$$Q_{\hat{n}} = e^{i \vec{\sigma}_n \cdot \frac{\alpha_n}{2}} \cong 1 + i \sigma_n \frac{\alpha_n}{2}$$

$$Q_z = \begin{pmatrix} e^{i \frac{\alpha_z}{2}} & \\ & e^{-i \frac{\alpha_z}{2}} \end{pmatrix}$$

$$QPQ^{\dagger} = \begin{pmatrix} z & e^{i\alpha}(x-iy) \\ e^{-i\alpha}(x+iy) & z \end{pmatrix}$$

$$e^{i\alpha}(x-iy) = (c \pm is)(x \mp iy)$$
$$= (cx + sy) + i(-sx + cy)$$

$$e^{-i\alpha}(x+iy) = $$

$$\left.\begin{aligned} x &\longrightarrow cx + sy \\ y &\longrightarrow -sx + cy \end{aligned}\right\} \longrightarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Euler's Theorem

For cont. Tx. $A \Rightarrow$ same as rot. about fixed axis. (For $n =$ odd)

$$X' = AX$$

$$R' = AR = R \quad \text{for special vector } R$$

$$AR = \lambda R \quad w/\ \lambda = 1$$

$$(A - \lambda \mathbb{1}) R = 0$$

$\Rightarrow$ Real orthog. matrix A: $(A^T = A^{-1})$ specifying physical motion $\Rightarrow \lambda = +1$

$$(A - 1) A^T = (1 - A^T) \qquad \text{For } AA^T = 1$$

$$|A - 1|\,|A^T| = |1 - A^T| \qquad |A| = |A^T| = +1 \text{ for proper orthog. rot}$$

$$|A - 1| = |1 - A|$$

$$|A - 1| = (-1)^n |A - 1|$$

only true if $|A - \mathbb{1}| = 0$

$$\Rightarrow \exists \lambda \text{ s.t. } |A - \lambda \mathbb{1}| = 0$$

$$|A| = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_2 (1) = 1$$

$$\therefore \lambda_1 \lambda_2 = 1$$

$\Rightarrow$ ~~$\lambda_{1,2} = \pm 1$ but~~

what is mag. of λ :

$$R^{\dagger\prime} R' = (AR)^{\dagger} (AR) = R^{\dagger} R$$

$$= \lambda^{*} R^{\dagger} \lambda R$$

$$\Rightarrow \lambda \lambda^{*} = 1 = |\lambda|^{2}$$

$\therefore$ $|\lambda_1| = 1 = |\lambda_2|$ $\qquad$ $\lambda_1 \lambda_2 = 1$

$$\Rightarrow \lambda_1 = e^{+i\phi} \qquad \lambda_2 = e^{-i\phi}$$

Consider rot about z.

$$A = \begin{pmatrix} c & s & \\ -s & c & \\ & & 1 \end{pmatrix}$$

$$\mathrm{Tr}\, A = 1 + 2c = \lambda_1 + \lambda_2 + \lambda_3$$

$$|A| = c^2 + s^2 = 1$$

If $\lambda_3 = 1$, $\quad \lambda_1 + \lambda_2 = 2c_\phi$

$$e^{i\phi} + e^{-i\phi} =$$

Infinitesimal Rotations:

$$A = 1 + \varepsilon$$

$$A^{-1} = 1 - \varepsilon = A^T$$

$$A A^{-1} = 1 - \varepsilon^2 \sim 1$$

$$\therefore \quad A^T = (1+\varepsilon)^T = 1 + \varepsilon^T = 1 - \varepsilon$$

$$\Rightarrow \quad \varepsilon = -\varepsilon^T$$

$$\Rightarrow \quad \varepsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ & 0 & d\Omega_1 \\ & & 0 \end{pmatrix}$$

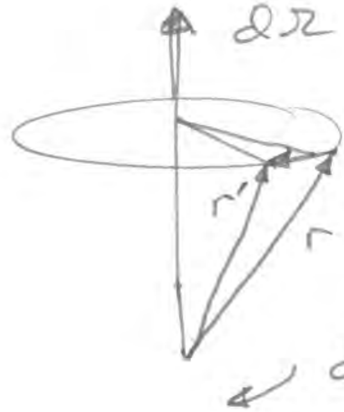
$$d\vec{x} = \vec{x}' - \vec{x} = A\vec{x} - \vec{x} = \varepsilon\vec{x} = \left(\vec{x} \times d\vec{\Omega}\right)$$

Try: $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad x_1' = x_1 + d\Omega_3 x_2 - d\Omega_2 x_3$

$$dx_1 = x_1' - x_1 = x_2 d\Omega_3 - x_3 d\Omega_2$$

$$\therefore \quad dx_i = \varepsilon_{ijk} x_j d\Omega_k$$

$$\boxed{d\vec{x} = \vec{x} \times d\vec{\Omega}}$$



$d\vec{r} = r' - r = \vec{r} \times d\vec{\Omega}$

$d\phi \sim d\Omega$

$$d\vec{\Omega} = \hat{n}\, d\phi$$

$C = D \times F \qquad C_i = \varepsilon_{ijk} D_j F_k$

Axial ~~Polar~~ vector   Invariant under parity

Coriolis Force:

$$(dG)_{space} = (dG)_{body} + (dG)_{Rot}$$

movement relative to body coords.

rotation of body

$$V_s = V_r + \omega \times r = \left(\frac{d}{dt} + \omega \times\right) r$$

space coords

Earth coords

rotation of earth

Take $\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \omega \times$

$\cancel{a_s = a_r +}$

$$\left(\frac{dV_s}{dt}\right)_s = \frac{dV_r}{dt} + \omega \times V_r + \frac{d(\omega \times r)}{dt} + \omega \times (\omega \times r)$$

$$a_s = a_r + 2(\omega \times v_r) + \omega \times (\omega \times r)$$

$F = m a_s \Rightarrow$

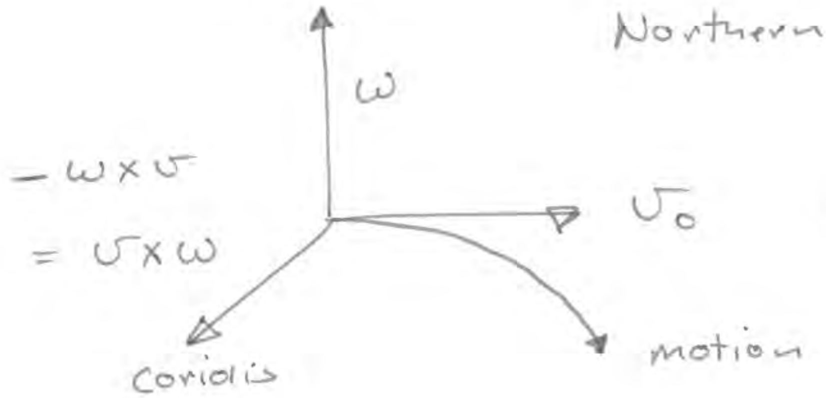
centrifugal

coriolis

stationary

$$m a_r = \underbrace{\underbrace{(m a_s)}_{F} - 2(m)(\omega \times v_r) - m\omega \times (\omega \times r)}_{F_{eff}}$$

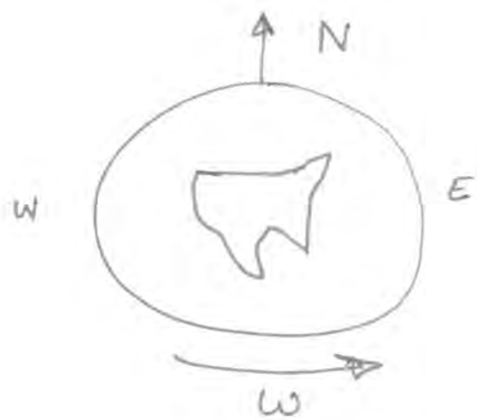
Northern hemisphere

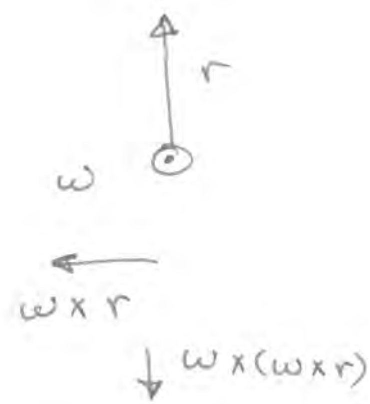


stationary: $|\omega \times (\omega \times r)| = m\omega^2 r \sin\theta$

$$= \frac{m v^2}{r} \sin\theta$$ centrifugal Force

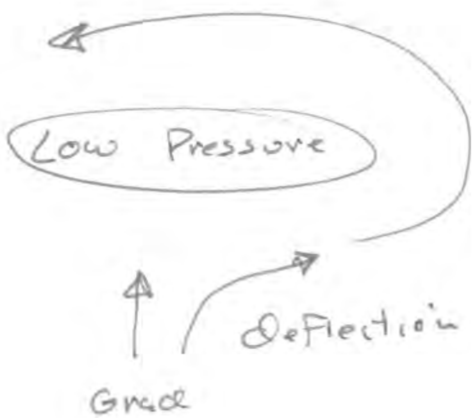
coriolis: $-2m(\omega \times v_r)$





Effective Force

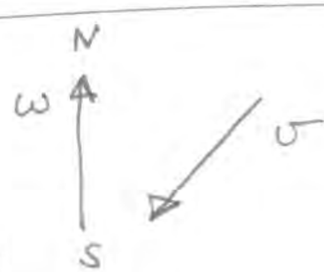
Winds



Circulation

in N Hemisphere

Free Fall



E $\otimes$

W $\odot$

$v \times \omega$ $\otimes$