$$S = \int dt\, L(q\dot{q})$$

$$\delta S = 0 = \int dt \left[\frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}\right]$$

$$= \int dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right] dq$$

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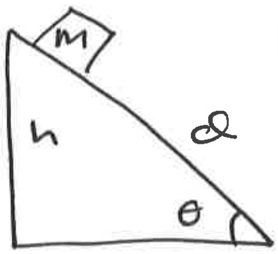



$h = d \sin\theta$ $\quad T = \frac{1}{2} m v^2$ $\quad V = mgh$

$$L = \frac{m}{2}\dot{x}^2 + mg\sin\theta\, x$$

$$\frac{\partial L}{\partial x} = -mg\sin\theta \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\rightarrow \quad m\ddot{x} = ma = mg\sin\theta$$

$$a = g\sin\theta$$

---

<u>Min Length</u> $\quad ds^2 = dx^2 + dy^2 = dx^2\left(1 + \frac{dy}{dx}^2\right)$

$$ds = dx\sqrt{1 + \dot{y}^2} = dx\, F(y\dot{y})$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial \dot{y}} = \text{const} \equiv \frac{2\dot{y}}{\sqrt{1+\dot{y}^2}} \Rightarrow \dot{y} = \text{const}$$

$$y = mx + b$$

# Brachistichrone


$$x = vt$$

$$t = \frac{x}{v} \qquad t^2 = \frac{ds^2}{v^2} = \frac{dx^2 + dy^2}{2gy}$$

$$\cancel{\tfrac{1}{2}} mgh = \frac{1}{2} m v^2$$

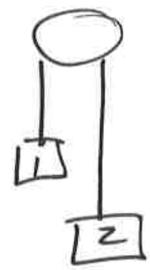
$$v^2 = 2gh$$

$$dt = \frac{dy\sqrt{1+\dot{x}^2}}{\sqrt{2gy}} \qquad \dot{x} = \frac{dx}{dy}$$

$$F(x, \dot{x}) = \sqrt{\frac{1+\dot{x}^2}{2gy}}$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial F}{\partial \dot{x}} = \text{const} = \frac{1}{2a} = \frac{2\dot{x}}{\sqrt{2gy}\sqrt{1+\dot{x}^2}}$$

# Atwood Machine



$$m_1 g - T = m_1 a$$

$$m_2 g - T = -m_2 a$$

$$a = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right) \qquad T = g\left(\frac{2 m_1 m_2}{m_1 + m_2}\right)$$

---

$$x_1 + x_2 = L \qquad x_2 = L - x_1$$

$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 \qquad V = m_1 g x_1 + m_2 g x_2$$

$$= \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_1^2 \qquad V = m_1 g x_1 + m_2 g (L - x_1)$$

$$L = T - V$$

$$\frac{\partial L}{\partial x_1} = m_1 g - m_2 g \qquad \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$$

$$\Longrightarrow \quad (m_1 + m_2)\ddot{x}_1 = g(m_1 - m_2) \quad \Longrightarrow \quad a = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

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# Lagrange Multipliers

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \lambda \frac{\partial F}{\partial q}$$

$$F = x_1 + x_2 - L = 0$$

$$\frac{\partial F}{\partial x_1} = 1 \qquad \frac{\partial F}{\partial x_2} = 1$$

Note $\ddot{x}_1 = -\ddot{x}_2$

$$\frac{\partial L}{\partial x_1} = m_1 g \qquad \frac{\partial L}{\partial x_2} = m_2 g \qquad \frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

① $m_1 g - m_1 \ddot{x}_1 = \lambda$

② $m_2 g - m_2 \ddot{x}_2 = \lambda$

with $\ddot{x}_1 = -\ddot{x}_2$

① − ② $\Rightarrow (m_1 - m_2) g - (m_1 + m_2)\ddot{x}_1 = 0$

$$\ddot{x}_1 = -\ddot{x}_2 = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$\lambda = \frac{2 m_1 m_2}{(m_1 + m_2)} g \approx \text{Tension}$$