Chpt 3

Kepler:

which $\Rightarrow F \sim 1/r^2$

1) Orbits are ellipses
2) Equal A w/ Equal Time
3) $\tau^2 \propto r^3$

Newton

$$F = \frac{G M m}{r^2}$$

Choose Polar Coords

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) \qquad V = ? \qquad \text{Assume } V = V(r)$$

$$\therefore \quad L = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - V(r)$$

Note: θ is absent $\Rightarrow P_\theta$ is conserved:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \equiv \frac{\partial}{\partial t} P_\theta \Rightarrow P_\theta = \text{const}$$

$$P_\theta = \boxed{m r^2 \dot{\theta} = \ell} = \text{const}$$

For r eq

$$m \ddot{r} - m r \dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0$$

$$\Rightarrow \boxed{m \ddot{r} - \frac{\ell^2}{m r^3} = -\frac{\partial V(r)}{\partial r}}$$

Use E - cons :

$$E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r)$$

$$\frac{dE}{dt} = 0 = \left[m\ddot{r} - \frac{\ell^2}{mr^3} + \frac{\partial V(r)}{\partial r} \right] \frac{\partial r}{\partial t}$$

$\hookrightarrow = 0$

$\Rightarrow$ $E = \text{const}$

$\therefore$ $$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r)}$$ 2nd Eq

---

Solve For $\dot{r}$

$$\dot{r}^2 = \frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2} \right)$$

Solve For $\dot{\theta}$

$$\dot{\theta} = \frac{\ell}{mr^2}$$

---

4 const. of Int: $\underbrace{E, \ell,}_{\text{Int. Consts.}} r_0, \theta_0$

Show $\dfrac{\partial E}{\partial t} = 0$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2m} \frac{1}{r^2} + V$$

Show $\dfrac{dE}{dt} = 0$

(1) $$\frac{d}{dt}\left(\frac{1}{2} m \dot{r}^2\right) = m \dot{r} \ddot{r}$$

(2) $$\frac{d}{dt}\left(\frac{\ell^2}{2m} r^{-2}\right) = -\frac{\ell^2}{m r^3} \dot{r}$$

(3) $$\frac{d}{dt}\left[V(r)\right] = \frac{dV}{dr} \frac{dr}{dt} = -F \dot{r}$$

$$\frac{dE}{dt} = \dot{r} \underbrace{\left[m \ddot{r} - \frac{\ell^2}{m r^3} - F \right]}_{\text{Eq of motion for } r}$$

$$\therefore \rightarrow 0 \quad \Rightarrow \quad \frac{dE}{dt} = 0$$

Sec 3-3 Effective 1-D Problem

Note: Eq. for r:

$$\dot{r}^2 = \frac{2}{m}\left(E - \left[V(r) + \frac{\ell^2}{2mr^2} \right] \right)$$

In analogue for a 1-D system

$$E = \frac{1}{2} m \dot{x}^2 + V(x)$$

$$\dot{x}^2 = \frac{2}{m} \left(E - V(x) \right)$$

∴ This is a 1-D problem w/ the replacement:

$$V(r) \longrightarrow V(r) + \frac{\ell^2}{2mr^2} = V(r) + \delta V(r)$$

Or, for the force: $F = -\nabla V$

$$\delta F = -\nabla(\delta V) = + \frac{\ell^2}{mr^3} = mr\dot{\theta}^2 = \frac{mv_\theta^2}{r}$$

centrifugal force

$$\therefore \quad F \longrightarrow F + \delta F = F + \frac{mv_\theta^2}{r}$$

we have chosen a non-inertial system

Likewise: $\hat{V}'(r)$

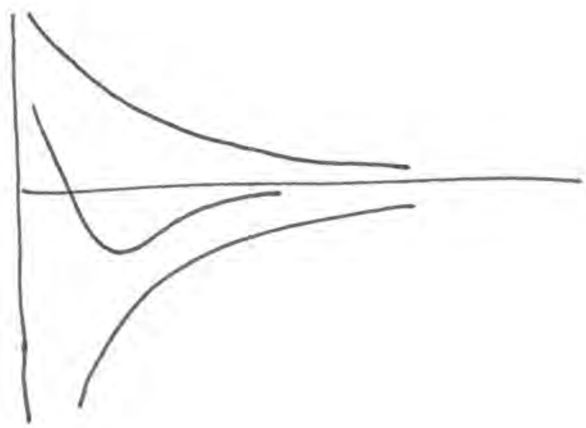
$$E \longrightarrow \frac{1}{2}m\dot{r}^2 + \left(V(r) + \delta V(r)\right)$$

Consider Kepler force

$$F = -\frac{k}{r^2} \qquad V = -\frac{k}{r}$$

$$\hat{V}' = -\frac{k}{r} + \frac{\ell^2}{2mr^2}$$

$E = T + V$



$E = 0$

Virial Theorem:

$$\mathring{p} = F \qquad G = p \cdot r \quad \left(\begin{array}{c}\text{No physical} \\ \text{Interpretation}\end{array}\right)$$

$$\frac{dG}{dt} = \frac{d}{dt}(p \cdot r) = \underbrace{\mathring{p} \cdot r}_{F \cdot r} + \underbrace{p \cdot \mathring{r}}_{2T = mv^2} = F \cdot r + 2T$$

Time Average

$$\frac{1}{\tau}\int_0^\tau dt \, \frac{dG}{dt} = \underbrace{\frac{1}{\tau}\left[G(\tau) - G(0)\right]}_{=0} = \overline{F \cdot r} + \overline{2T}$$

For periodic motion of period τ

$$\Rightarrow \quad \overline{T} = -\frac{1}{2}\overline{\sum F \cdot r} \qquad \left(\begin{array}{c}\text{Not True} \\ \text{w/out Aver}\end{array}\right)$$

For Cons. Forces: $F = -\nabla V = -\frac{\partial V}{\partial r}$

$$\Rightarrow \cancel{\overline{T} = +\frac{1}{2}\overline{V}}$$

For $V(r) = a\, r^{n+1} \Rightarrow \overline{T} = \frac{n+1}{2}\overline{V}$

$$V = -\frac{k}{r} \quad \underset{n=-2}{\Longrightarrow} \quad \overline{T} = -\frac{1}{2}\overline{V}$$

Sec 3-4:

Given

$$\dot{r}^2 = \frac{2}{m} T' \qquad \dot{\theta} = \frac{l}{mr^2}$$

$$T' = E - \left(V + \frac{l^2}{2mr^2}\right) \qquad \text{EFF KE}$$

Eliminate t variable:

$$\frac{d\theta}{dt} = \frac{l}{mr^2}$$

$$\therefore \quad \frac{d}{dt} = \frac{l}{mr^2} \frac{d}{d\theta}$$

$$\therefore \quad \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\left(\frac{l}{mr^2}\right) = \sqrt{\frac{2}{m} T'}$$

$$\frac{dr}{d\theta} = \sqrt{\frac{2mr^4}{l^2} T'} = \frac{mr^2}{l}\sqrt{\frac{2}{m} T'}$$

Closed Orbits :

$$V' = V + \frac{\ell^2}{2mr^2}$$

$$V'(r) = V'(r_0) + \left.\frac{\partial V'}{\partial r}\right|_{r_0}(r-r_0) + \frac{1}{2}\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r_0}(r-r_0)^2$$

Assume $\quad V = -\frac{K}{r^n} \qquad \frac{\partial V}{\partial r} = +\frac{nK}{r^{n+1}} \qquad \frac{\partial^2 V}{\partial r^2} = \frac{-n(n+1)K}{r^{n+2}}$

For min, $\left.\frac{\partial V'}{\partial r}\right|_{r_0} = 0 \Rightarrow \frac{nK}{r^{n+1}} = \frac{\ell^2}{mr^3}$

For bounded orbit. $\frac{\partial^2 V'}{\partial r^2} > 0 \Rightarrow$

$$-\frac{n(n+1)K}{r^{n+2}} + \frac{3\ell^2}{mr^4} > 0$$

$$\hookrightarrow \quad -(n+1)\frac{\ell^2}{mr^4} + 3\frac{\ell^2}{mr^4} > 0 \qquad \text{From } \left.\frac{\partial V'}{\partial r}\right| = 0$$

$$-(n+1) + 3 > 0$$

$$\boxed{2 > n}$$

Consider V' harmonic potential

$$V'(r) = V'(r_0) + \frac{1}{2} C (\Delta r)^2$$

$$C = \left.\frac{\partial^2 V'}{\partial r^2}\right|_{r_0} = \left[-(n+1)+3\right] \frac{l^2}{mr^4} = (2-n)\frac{l^2}{mr^4}$$

SHO $\quad T = \frac{1}{2} m\dot{r}^2 \quad V = \frac{1}{2} k r^2$

$$L = T - V$$

$$Eq \Rightarrow \quad m\ddot{r} + kr = 0$$

$$r(t) = r_{AMP}\, e^{i\omega t} \qquad \omega^2 = \frac{k}{m}$$

Shift by r_0

$$r(t) = r_0 + r_{AMP}\, e^{i\omega t}$$

$$= r_0 + r_{AMP} \cos \omega t$$

Need relation of t and θ

$$\dot{\theta} = \frac{\ell}{mr^2} \Rightarrow \theta = t\left(\frac{\ell}{mr^2}\right)$$

$$\therefore \quad \omega t = \sqrt{\frac{k}{m}}\left(\frac{mr^2}{\ell}\right)\theta$$

$$= \sqrt{(2-n)\frac{\ell^2}{mr^4}\cdot\frac{1}{m}\frac{m^2r^4}{\ell^2}}\;\theta$$

$$\omega t = \sqrt{(2-n)}\;\theta$$

$$r(t) = r_0 + r_{AMP}\cos\sqrt{2-n}\;\theta$$

Orbit is periodic if $\sqrt{2-n} = \text{rat} = \frac{p}{q}$

<u>Examples</u>

$V = -\frac{k}{r} \quad n=1 \quad \sqrt{2-n} = 1$

$V = -\frac{k}{r^{-2/9}} \quad n=\frac{2}{9} \quad \sqrt{2-\frac{2}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$

$V = \frac{1}{2}kr^2 \quad n=-2 \quad \sqrt{2-(-2)} = 4$

Kepler Problem

For $V = -\frac{k}{r}$

$$\frac{1}{r} = C(1 + e\cos\theta) \qquad C = \frac{mk}{\ell^2}$$

$$r = \frac{1}{C(1 + e\cos\theta)} \qquad e = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

Classify orbits in terms of e

$e < 1$ bounded

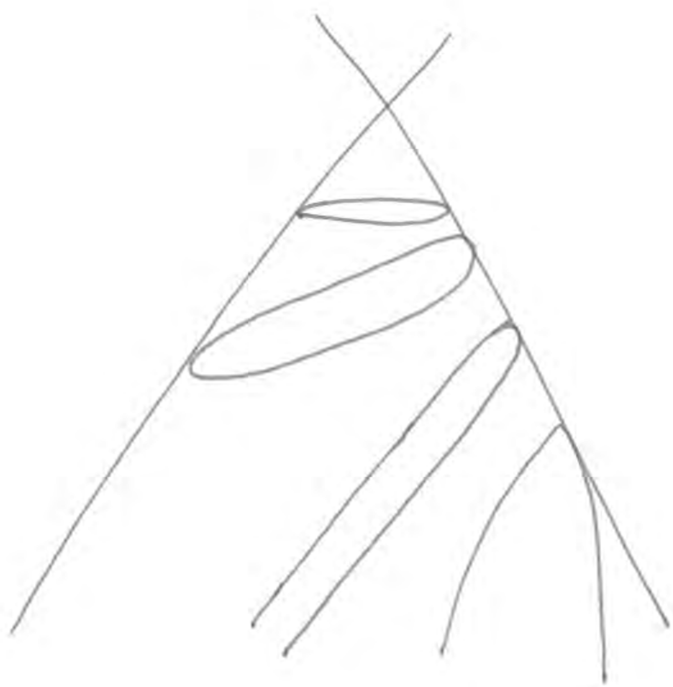
$e > 1$ unbounded $r \to \infty$

$e > 1$ $E > 0$ hyperbola

$e = 1$ $E = 0$ Parabola

$0 < e < 1$ $E < 0$ ellipse

$e = 0$ $E = -\frac{mk^2}{2\ell^2}$ circle



For $V = -k/r^n$ at min r_0 $\quad \frac{nk}{r^{n+1}} = \frac{\ell^2}{mr^3}$

$$\Rightarrow V(r_0) = -\frac{k}{r^n} = -\frac{1}{n}\frac{\ell^2}{mr^2}$$

For $n=1$ $\quad V(r_0) = \frac{-\ell^2}{mr^2} \equiv -\frac{k}{r_0}$

For circular orbit $\quad E = \frac{-mk^2}{2\ell^2} = -\frac{k}{2r_0} = \frac{V}{2}$

So $T = -\frac{V}{2}$ ala Virial Theorem

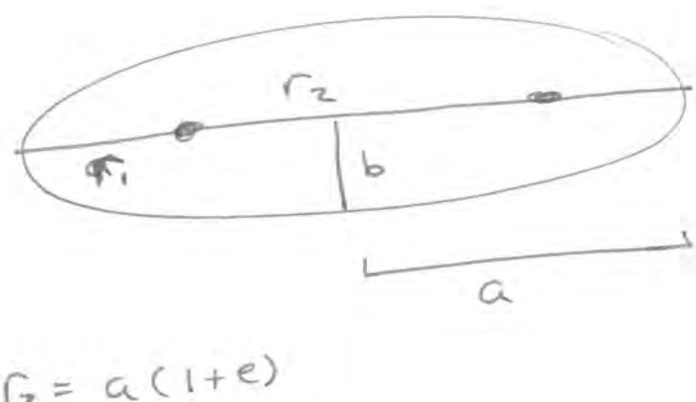
Turning points:

$$E = T - \frac{k}{r} + \frac{\ell^2}{2mr^2}$$

For $T=0$ $\hookrightarrow$ $r^2\left(\frac{E}{E}\right) + \left(\frac{k}{E}\right)r - \left(\frac{\ell^2}{2m}\right)\frac{1}{E} = 0$

$r_\pm$ = roots of $\nearrow$

$$a = \frac{r_1+r_2}{2} = +\frac{k}{2(-E)}$$



$r_2 = a(1+e)$

$r_1 = a(1-e)$

$r_1 + r_2 = 2a$