Exam 3 rules:

Open notes, open book, closed neighbor, closed internet. You may use Mathematica.
Exam is due before Thanksgiving.
Total points: 100. Each problem is 20 points.

PROBLEM 1:

1a) A spider is hanging by a silk thread from a tree in Dallas. Find the orientation and the value of the equilibrium angle that the thread makes with the vertical (i.e. with the direction of gravity), taking into account the rotation of the Earth. Assume that the latitude of Dallas is $\theta \approx 33$ and the radius of the Earth is $R \approx 6,400$km. [Important; note the spider is stationary. This should simplify the problem. Think.]

1b) In Dallas ($\theta \approx 33$), I shoot an arrow east with velocity v=50m/s. What is the magnitude of the Coriolis force compared to the gravitational force? What is the direction?

PROBLEM 2:

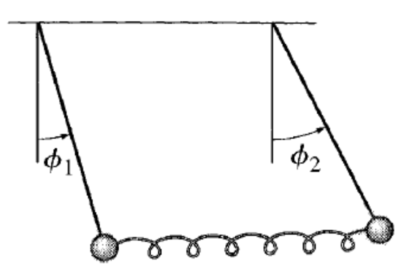
A rigid body consists of 4 point masses:

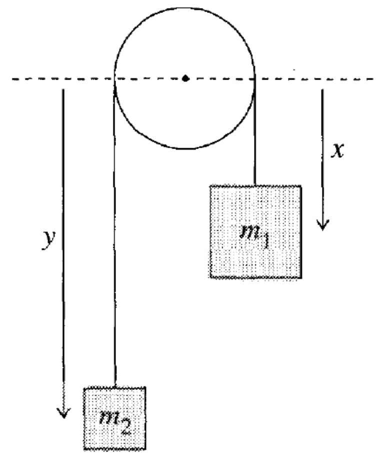
m at (a,0,0)
2m at (0,a,0)
3m at (0,a,a)
4m at (0,a,-a)

Find a) the moment of inertia tensor, b) the principle moments, and c) the orthogonal principle axes.
Note: I suggest you use Mathematica for part of this; but, make sure I can follow your notation!!!

PROBLEM 3:

Consider two identical plane pendulums (each of length L and mass m) that are joined by a massless spring (force constant k) as shown in Figure 11.17. The pendulums' positions are specified by the angles ϕ_1 and ϕ_2 shown. The natural length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical. **(a)** Write down the total kinetic energy and the gravitational and spring potential energies. [Assume that both angles remain small at all times. This means that the extension of the spring is well approximated by $L(\phi_2 - \phi_1)$.] Write down the Lagrange equations of motion. **(b)** Find and describe the normal modes for these two coupled pendulums.





PROBLEM 4:

Solve the Atwood machine shown at right using Hamilton's method. Use two variables {x,y}, and obtain Hamilton's equations for both x and y. Then use the constraint x+y=L to find the acceleration of the system and show this equals the expected result.

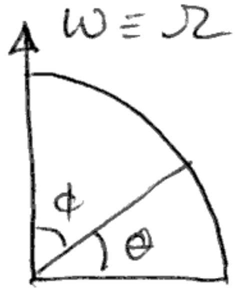
PROBLEM 5:

a) (This is modeled from Example 14.2 in the text.) 10^6 neutrons are fired through a gold foil 0.1mm thick. Take the gold nucleus cross section to be 98.7 barns. How many neutrons will be scattered. The specific gravity of gold is 19.32.

b) (This is modeled from Example 14.5 in the text.) Using what you learned in Example 14.5, find the fraction of the scattered neutrons that are scattered within 10 degrees of the source; that is between angles 170 and 190 degrees. [Think]

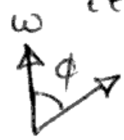
Exam #3

#1 A



$F_{coriolis} = 0 \qquad F_{cent} \neq 0$

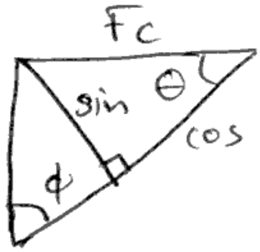
$F_{cent} = -m\, \omega \times (\omega \times r)$



$-\omega \times r = \otimes \quad \sin\phi\, \omega r$

$= \cos\theta\, \omega r$

$-\omega \times (\omega \times r) = \longrightarrow \omega^2 r \cos\theta$



Compone $\perp$ to string

$= F_c \sin\theta$

$F = F_c \sin\theta = m\omega^2 r \sin\theta \cos\theta$

$$\tan\alpha = \frac{\omega^2 r \sin\theta \cos\theta}{g} \Rightarrow \alpha = 0.09^\circ$$

B

$F_{coriolis} = +m\, v \times \omega \qquad F_{cent} = -m\omega \times (\omega \times r)$

$\longrightarrow \qquad\qquad \longrightarrow$

$\omega^2 r \cos\theta$

$m\omega v$

$$\text{ratio} = \frac{F_{cor}}{mg} = \frac{\omega v}{g} = = 0.00037$$

#3

$$T = \frac{1}{2} m L^2 \left(\dot{\phi}_1^2 + \dot{\phi}_2^2 \right)$$

$$V = \frac{1}{2} m g L \left(\phi_1^2 + \phi_2^2 \right) + \frac{1}{2} K L^2 \left(\phi_1 - \phi_2 \right)^2$$

Note: $1 - \cos\phi \cong \frac{1}{2}\phi^2$

$$T_{mat} = L^2 m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad V_{mat} = \begin{pmatrix} KL^2 + mgL & -KL^2 \\ -KL^2 & mgL + KL^2 \end{pmatrix}$$

$$\omega^2 = \frac{g}{L}, \quad \frac{g}{L} + \frac{2K}{m}$$

4) Hamiltonian

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad V = m_1 g x_1 + m_2 g x_2$$

$x_1 + x_2 = \text{Length}$ constraint $\Rightarrow \boxed{\dot{x}_1 = -\dot{x}_2}$

$$L = T - V + \lambda(x_1 + x_2)$$

$$H = \sum_i p_i \dot{q}_i - L \qquad p_i = \frac{\partial L}{\partial \dot{q}} = m_i x_i$$

$$\boxed{H = T + V - \lambda(x_1 + x_2)}$$

① $\dfrac{\partial H}{\partial x_1} = m_1 g - \lambda = -\dot{P}_1$ ② $\dfrac{\partial H}{\partial x_2} = m_2 g - \lambda = -\dot{P}_2$

③ $\dfrac{\partial H}{\partial P_1} = \dfrac{P_1}{m_1} = +\dot{x}_1$ ④ $\dfrac{\partial H}{\partial P_2} = \dfrac{P_2}{m_2} = +\dot{x}_2$

---

① − ② $\Rightarrow (m_1 - m_2) g = \dot{P}_2 - \dot{P}_1 = -\dot{P}_1 \dfrac{(m_1 + m_2)}{m_1}$

③, ④ $\Rightarrow \dot{x}_1 = -\dot{x}_2 \Rightarrow P_2 = -\dfrac{m_2}{m_1} P_1$

$$\Rightarrow \frac{\dot{P}_1}{m_1} = -\frac{(m_1 - m_2)}{(m_1 + m_2)} \equiv \ddot{x}_1 = -\ddot{x}_2$$

5

$$N_{sc} = N_{inc}\, n_{tar}\, \sigma$$

$N_{inc} = 10^{6}$ 197 AMU

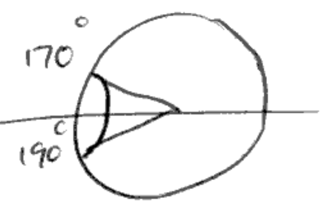
$\sigma = 98.7 \times 10^{-28}\ m^2$

$$n_{tar} = \frac{\rho t}{m} = \frac{(19.32 \times 10^{3}\ \frac{kg}{m^3})(10^{-4}\ m)}{197 \times 1.66 \times 10^{-27}\ kg} = 5.88 \times 10^{24}$$

$N_{sc} = 58{,}000$

---

b fraction



$$\frac{d\Omega}{4\pi} = 0.0075$$
$= 0.75\%$
$= 441$ neutrons

```
Needs["PhysicalConstants`"]
```

Problem 1:

Part a

```
Clear["Global`*"]
```

$$w = \frac{2\,\pi\,\text{rad}}{\text{day}} * \frac{1\,\text{day}}{24\,\text{hour}} * \frac{1\,\text{hour}}{60^2\,\text{sec}} \;//\; N$$

$$\frac{0.0000727221\,\text{rad}}{\text{sec}}$$

```
w = w /. {rad → 1, sec → 1}
```

0.0000727221

```
(* Needs["PhysicalConstants`"] *)
EarthRadius
```

6 378 140 Meter

$$r = \frac{\text{EarthRadius}}{\text{Meter}} \;//\; N$$

6.37814×10^{6}

```
angle = 33 Degree;
g = 9.8;
```

$$\text{ratio} = \frac{w^2\ r\,\text{Sin}[\text{angle}]\,\text{Cos}[\text{angle}]}{g}$$

0.00157217

$$\frac{\text{ArcTan}[\text{ratio}]}{\text{Degree}}$$

0.0900788

Part b

```
v = 50;
```

$$\text{ratio} = \frac{w\,v}{g} \;//\; \text{ScientificForm}$$

3.71031×10^{-4}

Problem 2:

PROBLEM 2 : A rigid body consists of 4 point masses :

m at (a, 0, 0)

2 m at (0, a, 0)
3 m at (0, a, a)
4 m at (0, a, -a)
Find a) the moment of inertia tensor, b) the principle moments, and c) the orthogonal principle axes.
Note: I suggest you use Mathematica for part of this; but, make sure I can follow your notation!!!

```
Clear["Global`*"]

location = {
   {a, 0, 0},
   {0, a, 0},
   {0, a, a},
   {0, a, -a}};
mass = {1, 2, 3, 4};
v = {vx, vy, vz} = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
```

Note, we program diagonal and off - diagonal with different formulas below!!!

```
Clear[term]
term[n_, i_, i_] :=
 +mass[[n]] (location[[n]].location[[n]] - (location[[n]].v[[i]])^2)
term[n_, i_, j_] := -mass[[n]] (location[[n]].v[[i]] * location[[n]].v[[j]])

mat = Table[ Sum[ term[n, i, j], {n, 1, 4}], {i, 1, 3}, {j, 1, 3}];
mat // MatrixForm
```

$$\begin{pmatrix} 16\,a^2 & 0 & 0 \\ 0 & 8\,a^2 & a^2 \\ 0 & a^2 & 10\,a^2 \end{pmatrix}$$

```
evec = (Normalize /@ Eigenvectors[mat]) // Simplify // Transpose ;
evec // MatrixForm (* Column form *)
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & -\frac{1+\sqrt{2}}{\sqrt{2\left(2+\sqrt{2}\right)}} \\ 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{2\left(2+\sqrt{2}\right)}} \end{pmatrix}$$

```
evec[[2]]
```

$$\left\{0,\ \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}},\ -\frac{1+\sqrt{2}}{\sqrt{2\left(2+\sqrt{2}\right)}}\right\}$$

```
eval = Eigenvalues[mat] ;
eval // Simplify // TableForm
```

$16\ a^2$

$(9+\sqrt{2})\ a^2$

$-(-9+\sqrt{2})\ a^2$

```
Transpose[evec] .mat. evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 16\ a^2 & 0 & 0 \\ 0 & \frac{(-16+7\sqrt{2})\ a^2}{-2+\sqrt{2}} & 0 \\ 0 & 0 & \frac{(16+7\sqrt{2})\ a^2}{2+\sqrt{2}} \end{pmatrix}$$

```
Transpose[evec] . evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3:

PROBLEM 3

```
Clear["Global`*"]
```

$$T = \frac{m}{2} L^2 \left(d\phi1^2 + d\phi2^2\right);$$

$$V = \frac{m}{2} g\, L \left(\phi1^2 + \phi2^2\right) + \frac{k}{2} L^2 (\phi1 - \phi2)^2;$$

```
T // Expand
```

$$\frac{1}{2} d\phi1^2\ L^2\ m + \frac{1}{2} d\phi2^2\ L^2\ m$$

```
V // Expand
```

$$\frac{1}{2} k\ L^2\ \phi1^2 + \frac{1}{2} g\ L\ m\ \phi1^2 - k\ L^2\ \phi1\ \phi2 + \frac{1}{2} k\ L^2\ \phi2^2 + \frac{1}{2} g\ L\ m\ \phi2^2$$

```
D[T, dϕ1]
```

$d\phi1\ L^2\ m$

```
Tmat = m L^2 {{1, 0}, {0, 1}};
Tmat // MatrixForm
```

$$\begin{pmatrix} L^2\ m & 0 \\ 0 & L^2\ m \end{pmatrix}$$

```
D[V, ϕ1] // Expand
```

$k\ L^2\ \phi1 + g\ L\ m\ \phi1 - k\ L^2\ \phi2$

```
D[V, ϕ2] // Expand
```

$-k\ L^2\ \phi1 + k\ L^2\ \phi2 + g\ L\ m\ \phi2$

```
Vmat = {{k L^2 + g L m, -k L^2}, {-k L^2, k L^2 + g L m}};
Vmat // MatrixForm
```

$$\begin{pmatrix} k L^2 + g L m & -k L^2 \\ -k L^2 & k L^2 + g L m \end{pmatrix}$$

```
mat = Vmat - w2 Tmat;
mat // MatrixForm
```

$$\begin{pmatrix} k L^2 + g L m - L^2 m w2 & -k L^2 \\ -k L^2 & k L^2 + g L m - L^2 m w2 \end{pmatrix}$$

```
sol0 = Solve[Det[mat] == 0, w2] // Simplify
```

$$\left\{\left\{w2 \to \frac{g}{L}\right\}, \left\{w2 \to \frac{g}{L} + \frac{2 k}{m}\right\}\right\}$$

```
mat /. sol0[[1]]
```

$$\left\{\left\{k L^2, -k L^2\right\}, \left\{-k L^2, k L^2\right\}\right\}$$

```
eq1 = (mat /. sol0[[1]]).{a, b} == {0, 0} // Thread
```

$$\left\{a k L^2 - b k L^2 == 0, -a k L^2 + b k L^2 == 0\right\}$$

```
norm = a^2 + b^2 == 1
```

$$a^2 + b^2 == 1$$

```
sol1 = Solve[Join[eq1, {norm}], {a, b}]
```

$$\left\{\left\{a \to -\frac{1}{\sqrt{2}}, b \to -\frac{1}{\sqrt{2}}\right\}, \left\{a \to \frac{1}{\sqrt{2}}, b \to \frac{1}{\sqrt{2}}\right\}\right\}$$

```
eVec1 = {a, b} /. sol1[[2]]
```

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

```
mat /. sol0[[2]]
```

$$\left\{\left\{k L^2 + g L m - L^2 \left(\frac{g}{L} + \frac{2 k}{m}\right) m, -k L^2\right\}, \left\{-k L^2, k L^2 + g L m - L^2 \left(\frac{g}{L} + \frac{2 k}{m}\right) m\right\}\right\}$$

```
eq2 = (mat /. sol0[[2]]).{a, b} == {0, 0} // Thread
```

$$\left\{-b k L^2 + a \left(k L^2 + g L m - L^2 \left(\frac{g}{L} + \frac{2 k}{m}\right) m\right) == 0, -a k L^2 + b \left(k L^2 + g L m - L^2 \left(\frac{g}{L} + \frac{2 k}{m}\right) m\right) == 0\right\}$$

```
norm = a^2 + b^2 == 1
```

$$a^2 + b^2 == 1$$

```
sol2 = Solve[Join[eq2, {norm}], {a, b}]
```

$$\left\{\left\{a \to -\frac{1}{\sqrt{2}}, b \to \frac{1}{\sqrt{2}}\right\}, \left\{a \to \frac{1}{\sqrt{2}}, b \to -\frac{1}{\sqrt{2}}\right\}\right\}$$

```
eVec2 = {a, b} /. sol2[[2]]
```

$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$

```
evec = {eVec1, eVec2};
evec // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
Transpose[evec] . evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
evec.Tmat.Transpose[evec] // Simplify // MatrixForm
```

$$\begin{pmatrix} L^2 m & 0 \\ 0 & L^2 m \end{pmatrix}$$

```
evec.Vmat.Transpose[evec] // Simplify // MatrixForm
```

$$\begin{pmatrix} g L m & 0 \\ 0 & L (2 k L + g m) \end{pmatrix}$$

Problem 5:

PROBLEM 5:
a) (This is modeled from Example 14.2 in the text.) 10^6 neutrons are fired through a gold foil 0.1mm thick. Take the gold nucleus cross section to be 98.7 barns. How many neutrons will be scattered. The specific gravity of gold is 19.32.

b) (This is modeled from Example 14.5 in the text.) Using what you learned in Example 14.5, find the fraction of the scattered neutrons that are scattered within 10 degrees of the source; that is between angles 170 and 190 degrees. [Think]

AMU of Gold is 197

```
Clear["Global`*"]
```

Part a)

```
nIncident = 10^6;
thick = 10^-4;
sig = 98.7 * 10^-28;
density = 19.32 * 10^3;
mass = 197 * 1.667 * 10^-27;
```

```
nTarget = (density thick)/mass
```

5.88309×10^{24}

```
nScattered = nIncident nTarget sig
```

58 066.1

Part b)

Assuming the gold nucleus acts as a hard sphere, then the scattering is isotropic. [This is not correct, but I' ll save the more difficult Rutherford scattering problem for the homework.] We thus need to find the solid angle that covers the 10-degree cone around the backward region. The total cross section is given by:

$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{\pi} \text{Sin}[\theta]\, d\theta \int_0^{2\pi} d\phi$

We just want the part from 170-degrees to 180-degrees:

$\int_{\theta_{170}}^{\theta_{180}} \text{Sin}[\theta]\, d\theta \int_0^{2\pi} d\phi$

```
π/Degree // N
```

180.

```
180 Degree // N
```

3.14159

```
θ1 = 170 Degree
```

170 °

```
int[θ_] = Integrate[Sin[θ], {θ, θ, π}, {ϕ, 0, 2 π}]
```

$2\pi\,(1 + \text{Cos}[\theta])$

```
int[0]
```

4π

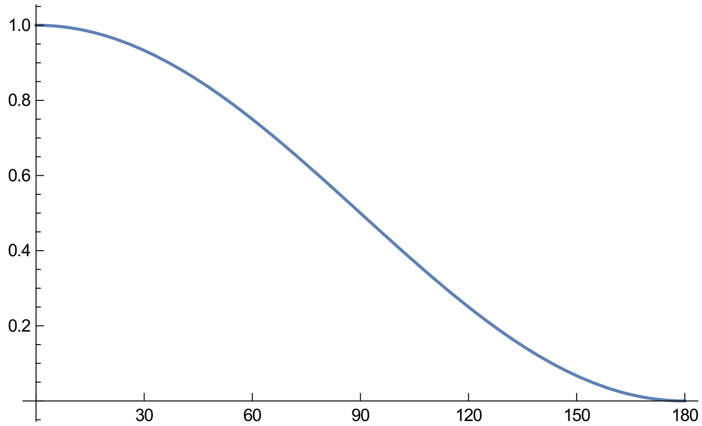
```
int[π]
```

0

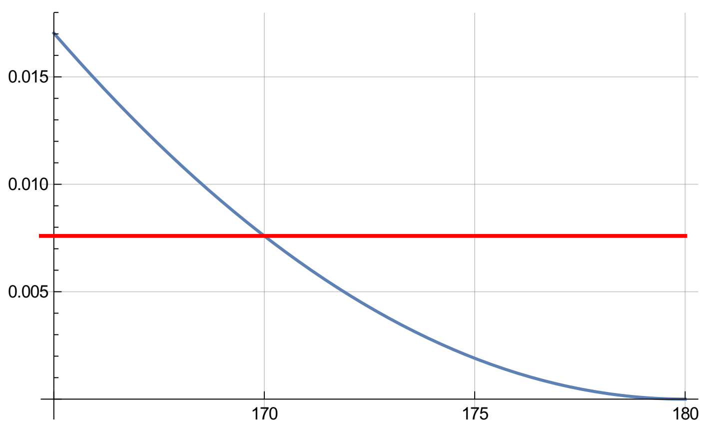
```
value170 = int[170. Degree]/(4 π)
```

0.00759612

```
Plot[int[θ Degree]/(4 π), {θ, 0, 180},
 Ticks → {Table[i, {i, 0, 180, 30}], Automatic}]
```




```
tab = Table[i, {i, 150, 180, 5}];
Plot[int[θ Degree]/(4 π), {θ, 165, 180},
 Ticks → {tab, Automatic},
 GridLines → {tab, Automatic},
 Epilog → {Red, Thick, Line[{{0, value170}, {180, value170}}]}
]
```



$$\texttt{Table}\left[\left\{\theta, \frac{\texttt{int}[\theta\,\texttt{Degree}]}{4.\,\pi}, \frac{\texttt{int}[\theta\,\texttt{Degree}]}{4.\,\pi}\,\texttt{nScattered}\right\}, \{\theta, 0, 180, 10\}\right] \;\texttt{//}\; \texttt{TableForm}$$

0	1.	58066.1
10	0.992404	57625.
20	0.969846	56315.2
30	0.933013	54176.4
40	0.883022	51273.6
50	0.821394	47695.1
60	0.75	43549.6
70	0.67101	38962.9
80	0.586824	34074.6
90	0.5	29033.
100	0.413176	23991.5
110	0.32899	19103.2
120	0.25	14516.5
130	0.178606	10371.
140	0.116978	6792.44
150	0.0669873	3889.69
160	0.0301537	1750.91
170	0.00759612	441.077
180	0.	0.