10.35 ★★ A rigid body consists of three masses fastened as follows: m at $(a, 0, 0)$, $2m$ at $(0, a, a)$, and $3m$ at $(0, a, -a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

10.36 ★★ A rigid body consists of three equal masses (m) fastened at the positions $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

Problem 1:

10.35 ★★ A rigid body consists of three masses fastened as follows: m at $(a, 0, 0)$, $2m$ at $(0, a, a)$, and $3m$ at $(0, a, -a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

10.36 ★★ A rigid body consists of three equal masses (m) fastened at the positions $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

```
Clear["Global`*"]
```

Three equal mass points are located at (a,0,0), (0, a, 2a) and (0, 2a, a). Find the principal moments of inertia about the origin and a set of principal axes.

```
location = {
   {a, 0, 0},
   {0, a, a},
   {0, a, -a}};
mass = {1, 2, 3};
v = {vx, vy, vz} = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
```

Note, we program diagonal and off - diagonal with different formulas below!!!

```
Clear[term]
term[n_, i_, i_] :=
 +mass[[n]] (location[[n]].location[[n]] - (location[[n]].v[[i]])^2)
term[n_, i_, j_] := -mass[[n]] (location[[n]].v[[i]] * location[[n]].v[[j]])
```

```
mat = Table[ Sum[ term[n, i, j], {n, 1, 3}], {i, 1, 3}, {j, 1, 3}];
mat // MatrixForm
```

$$\begin{pmatrix} 10\,a^2 & 0 & 0 \\ 0 & 6\,a^2 & a^2 \\ 0 & a^2 & 6\,a^2 \end{pmatrix}$$

```
evec = (Normalize /@ Eigenvectors[mat] ) // Simplify // Transpose ;
evec // MatrixForm (* Column form *)
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
evec[[2]]
```

$$\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

```
eval = Eigenvalues[mat] ;
eval // Simplify // TableForm
```

$10\,a^2$
$7\,a^2$
$5\,a^2$

```
Transpose[evec] .mat. evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 10\,a^2 & 0 & 0 \\ 0 & 7\,a^2 & 0 \\ 0 & 0 & 5\,a^2 \end{pmatrix}$$

```
Transpose[evec] . evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2:

10.35 ★★ A rigid body consists of three masses fastened as follows: m at $(a, 0, 0)$, $2m$ at $(0, a, a)$, and $3m$ at $(0, a, -a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

10.36 ★★ A rigid body consists of three equal masses (m) fastened at the positions $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. **(a)** Find the inertia tensor **I**. **(b)** Find the principal moments and a set of orthogonal principal axes.

```
Clear["Global`*"]
```

Three equal mass points are located at (a,0,0), (0, a, 2a) and (0, 2a, a). Find the principal moments of inertia about the origin and a set of principal axes.

```
location = {
    {a, 0, 0},
    {0, a, 2 a},
    {0, 2 a, a}};
mass = {1, 1, 1};
v = {vx, vy, vz} = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}};
```

Note, we program diagonal and off - diagonal with different formulas below!!!

```
Clear[term]
term[n_, i_, i_] :=
 +mass[[n]] (location[[n]].location[[n]] - (location[[n]].v[[i]])^2)
term[n_, i_, j_] := -mass[[n]] (location[[n]].v[[i]] * location[[n]].v[[j]])
```

```
mat = Table[ Sum[ term[n, i, j], {n, 1, 3}], {i, 1, 3}, {j, 1, 3}];
mat // MatrixForm
```

$$\begin{pmatrix} 10\,a^2 & 0 & 0 \\ 0 & 6\,a^2 & -4\,a^2 \\ 0 & -4\,a^2 & 6\,a^2 \end{pmatrix}$$

```
evec = (Normalize /@ Eigenvectors[mat] ) // Simplify // Transpose ;
evec // MatrixForm (* Column form *)
```

$$\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
evec[[2]]
```

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
eval = Eigenvalues[mat] ;
eval // Simplify // TableForm
```

$10\,a^2$
$10\,a^2$
$2\,a^2$

```
Transpose[evec] .mat. evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 10\,a^2 & 0 & 0 \\ 0 & 10\,a^2 & 0 \\ 0 & 0 & 2\,a^2 \end{pmatrix}$$

```
Transpose[evec] . evec // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3:

```
Clear["Global`*"]
```

```
m = {{1/Sqrt[2], 1/Sqrt[2], 0}, {0, 0, 1}, {1/Sqrt[2], -1/Sqrt[2], 0}};
m // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

```
Det[m]
```

1

```
Tr[m]
```

$$\frac{1}{\sqrt{2}}$$

```
eval = Eigenvalues[m] // Simplify
```

$$\left\{\frac{1}{4}\left(-2+\sqrt{2}+i\sqrt{10+4\sqrt{2}}\right), \frac{1}{4}\left(-2+\sqrt{2}-i\sqrt{10+4\sqrt{2}}\right), 1\right\}$$

```
Eigenvectors[m] // Simplify
```

$$\left\{\left\{\frac{1}{2}\left(1-\sqrt{2}+i\sqrt{5+2\sqrt{2}}-\frac{8i}{2i-i\sqrt{2}+\sqrt{10+4\sqrt{2}}}\right), -\frac{4i}{-i\left(-2+\sqrt{2}\right)+\sqrt{10+4\sqrt{2}}}, 1\right\},\right.$$
$$\left\{\frac{1}{2}\left(1-\sqrt{2}-i\sqrt{5+2\sqrt{2}}+\frac{8i}{i\left(-2+\sqrt{2}\right)+\sqrt{10+4\sqrt{2}}}\right), \frac{4i}{i\left(-2+\sqrt{2}\right)+\sqrt{10+4\sqrt{2}}}, 1\right\},$$
$$\left.\left\{1+\sqrt{2}, 1, 1\right\}\right\}$$

```
eq = Tr[m] == 1 + 2 Cos[θ]
```

$$\frac{1}{\sqrt{2}} == 1 + 2\,\mathrm{Cos}[\theta]$$

```
sol = Solve[eq, θ] /. {C[_] :> 0} // Normal
```

$$\left\{\left\{\theta \to -\mathrm{ArcCos}\left[\frac{1}{4}\left(-2+\sqrt{2}\right)\right]\right\}, \left\{\theta \to \mathrm{ArcCos}\left[\frac{1}{4}\left(-2+\sqrt{2}\right)\right]\right\}\right\}$$

```
θ / Degree /. sol // N
```

{-98.4211, 98.4211}

```
ev3 = Eigenvectors[m][[3]]
```

$$\left\{1+\sqrt{2}, 1, 1\right\}$$

```
m.ev3 // Simplify
```

$$\left\{1+\sqrt{2}, 1, 1\right\}$$

```
ev3n = Normalize[ev3] // Simplify
```

$$\left\{\frac{1+\sqrt{2}}{\sqrt{5+2\sqrt{2}}}, \frac{1}{\sqrt{5+2\sqrt{2}}}, \frac{1}{\sqrt{5+2\sqrt{2}}}\right\}$$

```
ev3n // N
```

{0.862856, 0.357407, 0.357407}

```
ev3n.{1, 0, 0}
```

$$\frac{1+\sqrt{2}}{\sqrt{5+2\sqrt{2}}}$$

Problem 4:

```
Clear["Global`*"]
```

```
evals = {1, (Sqrt[3] + I) / 2, (Sqrt[3] - I) / 2}
```

$$\left\{1, \frac{1}{2}\left(i+\sqrt{3}\right), \frac{1}{2}\left(-i+\sqrt{3}\right)\right\}$$

```
trace = Sum[evals[[i]], {i, 1, 3}] // Expand
```

$1+\sqrt{3}$

```
eq = trace == 1 + 2 Cos[θ]
```

$1+\sqrt{3} == 1+2\,\mathrm{Cos}[\theta]$

```
sol = Solve[eq, θ] /. {C[_] :> 0} // Normal
```

$$\left\{\left\{\theta \to -\frac{\pi}{6}\right\}, \left\{\theta \to \frac{\pi}{6}\right\}\right\}$$

```
θ / Degree /. sol // N
```

{-30., 30.}