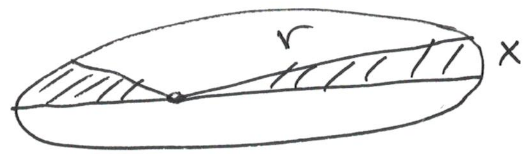
Kepler's 3 Laws

1) Conic Section: See graphics

2) Equal Area, Equal Time



$F = ma = \dot{P}$

$\tau = I\alpha = \dot{L} = 0 \Rightarrow L = \text{constant}$

$L = r \times P = rmv = rm\dot{X} = \text{constant}$

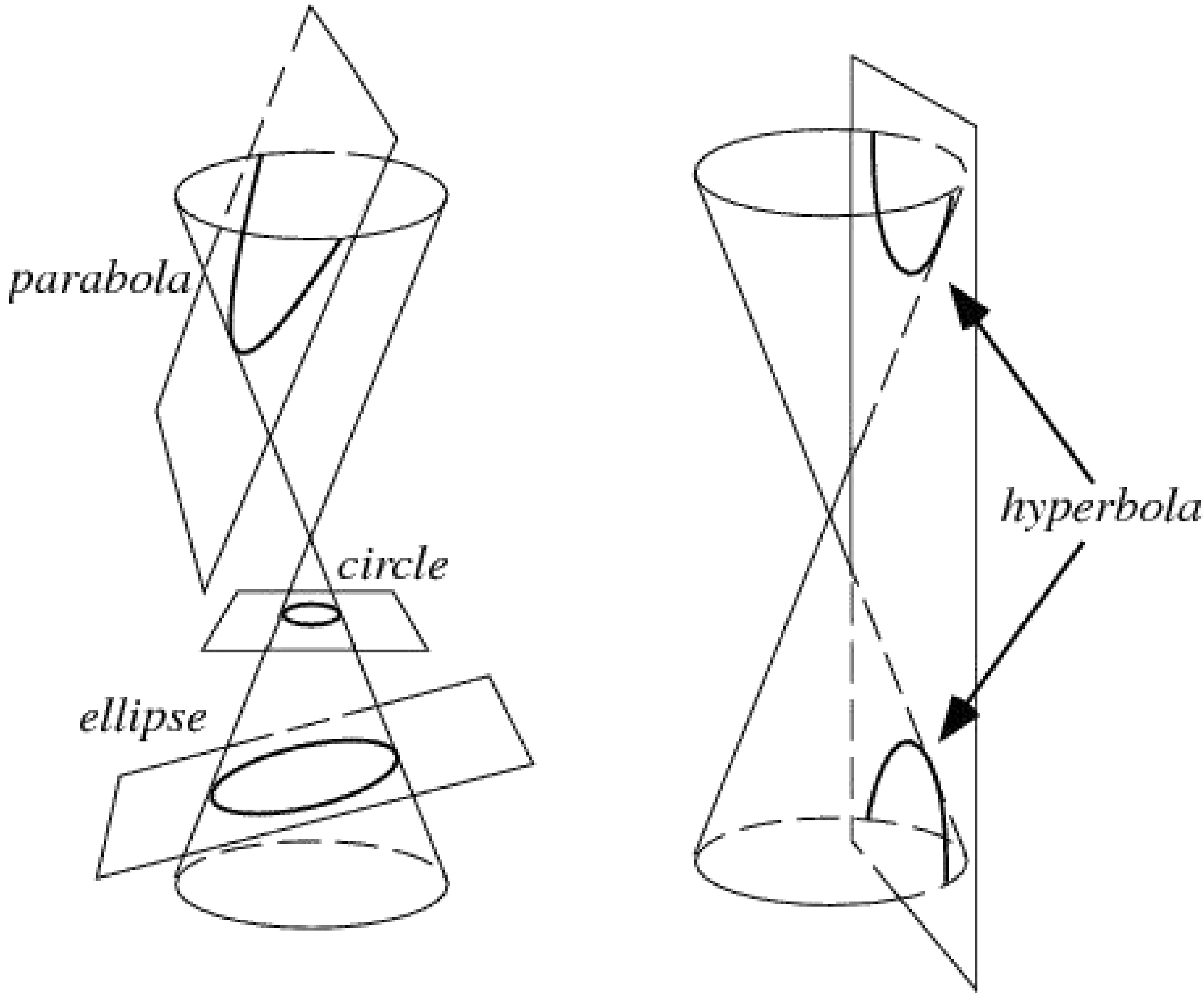
$A = \frac{1}{2} r X \qquad \dot{A} = \frac{1}{2} r \dot{X} \overset{\propto}{=} \text{constant}$

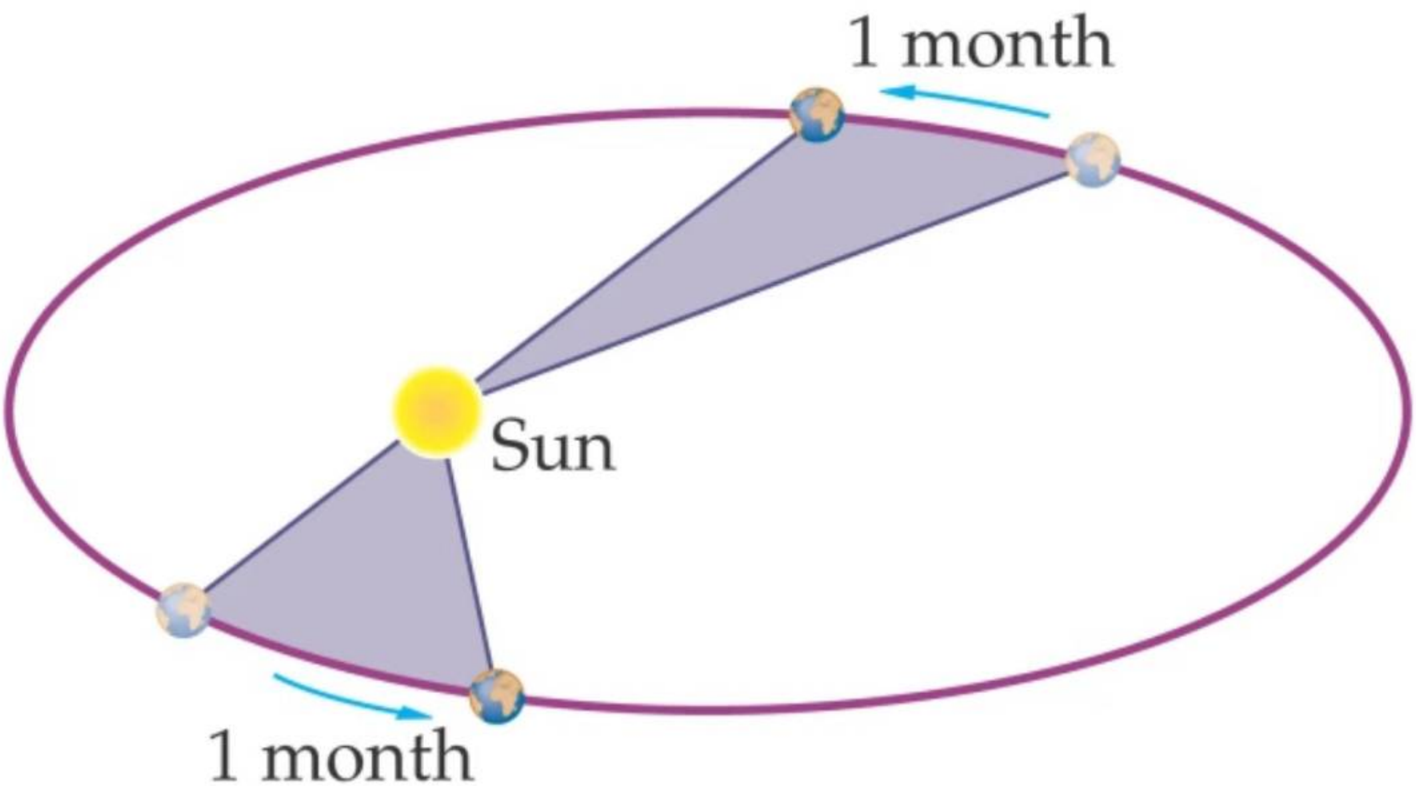
3) $T^2 \sim r^3$:

$F = ma \Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$

$\frac{GM}{r} = v^2 \qquad v = \frac{dx}{dt} = \frac{2\pi r}{T}$

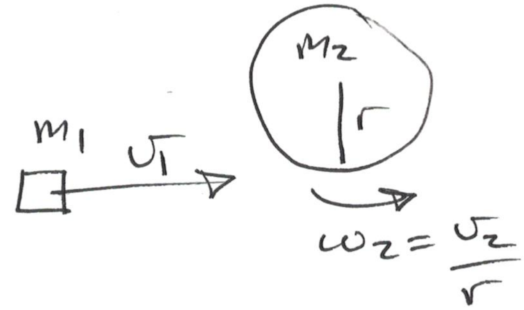
$\Rightarrow T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$


parabola
circle
ellipse
hyperbola


1 month
Sun
1 month

#2 Conserve Angular Momentum

$L_{before} = L_{after}$

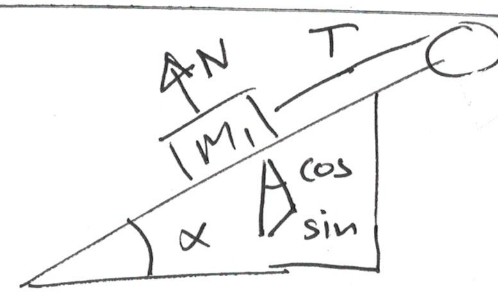


$r \times P = r m_1 v_1 = I \omega_2$

$r m_1 v_1 = \left(\frac{1}{2} m_2 + m_1\right) r^2 \cdot \frac{v_2}{r} = r v_2 \left(\frac{m_2}{2} + m_1\right)$

$v_2 = v_1 \left(\frac{m_1}{m_1 + m_2/2}\right) = r \omega_2$

#3 $F = ma$ $\quad \tau = I \alpha$



① $N = m_1 g \cos\theta$

② $m_1 g \sin\theta - T = m_1 a$

$\tau = r \times F \qquad I = \frac{1}{2} m_2 r^2 \qquad \alpha = \frac{a}{r}$

$\tau = I\alpha \Rightarrow \quad rT = \left(\frac{1}{2} m_2 r^2\right)\left(\frac{a}{r}\right)$

③ $T = \frac{1}{2} m_2 a$

unknowns $\{N, T, a\}$

$N = m_1 g \cos\theta$

$a = \frac{m_1 g \sin\theta}{m_1 + m_2/2}$

$T = \frac{m_1 m_2 g \sin\theta}{2m_1 + m_2}$

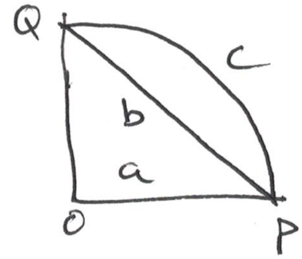
Limits: $m_2 = 0$

$a = g\sin\theta \qquad T = 0$

$m_1 = 0$

$a = 0 \qquad T = 0$

Problem 4.3 $F = (-y, x) = (-y)\hat{x} + (x)\hat{y}$



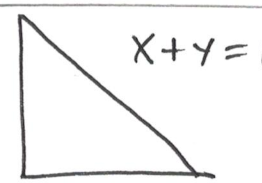
(a)

$$W = \int \vec{F}\cdot d\vec{r} = \int_P^O + \int_O^Q =$$

$$= \int_1^0 F_x(x,0)\,dx + \int_0^1 F_y(0,y)\,dy = 0 + 0 = 0$$

(b)

$$y = 1 - x \qquad dy = -dx$$



$$W = \int F_x(x,\underbrace{y}_{1-x})\,dx + F_y(x,\underbrace{y}_{1-x})\,\underbrace{dy}_{-dx}$$

$$= \int_1^0 -(1-x)\,dx + (x)(-dx) = \int_1^0 dx\,(-1)$$

$$= -x\Big]_1^0 = [0 - (-1)] = +1$$

(c)

$$(x,y) = (\cos\theta, \sin\theta)$$

$$(dx, dy) = (-\sin\theta, \cos\theta)\,d\theta$$

$$W = \int_C F_x\,dx + F_y\,dy = \int (-y)\,dx + (x)\,dy$$

$$= \int (-\sin\theta)(-\sin\theta\,d\theta) + (\cos\theta)(\cos\theta\,d\theta)$$

$$= \int_0^{\pi/2} (\sin^2\theta + \cos^2\theta)\,d\theta = \int_0^{\pi/2} d\theta = \theta\Big]_0^{\pi/2} = \frac{\pi}{2}$$

Time for cannon ball to fall from rest at lunar orbit distance to Earth, with $1/r^2$ gravitational acceleration.

Energy Conservation

M = Earth mass
m = cannon ball mass

Before

After

$$U_i + K_i = U_f + K_f$$

Assume Earth does not move.

$$-\frac{GMm}{r_0} + 0 = -\frac{GMm}{r} + \frac{1}{2} m \left[v(r)\right]^2$$

m cancels.

$$\Rightarrow v(r) = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} = -\frac{dr}{dt}$$

Separate variables.

$$\Rightarrow dt = -\frac{dr}{\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{r_0}}} = \frac{-dr}{\sqrt{2GM}} \sqrt{\frac{r r_0}{r_0 - r}}$$

Integrate both sides.

$$\int_0^T dt = -\frac{1}{\sqrt{2GM}} \int_{r_0}^{0} dr \sqrt{\frac{r r_0}{r_0 - r}}$$

$$T = \frac{1}{\sqrt{2GM}} \frac{\pi}{2} r_0^{3/2}$$

$$= \frac{1}{\sqrt{2\left(6.67\times10^{-11} \frac{Nm^2}{kg^2}\right)\left(5.972\times10^{24} kg\right)}} \frac{\pi}{2}\left(3.85\times10^{8} m\right)^{3/2}$$

$$= 4.20\times10^{5} s = 116.8 h = 4.86 d$$

Should use a different $g \sim r$ from the Earth's surface to the center.