In[1]:=
```
$Version
```
Out[1]=
```
12.1.0 for Linux x86 (64-bit) (March 14, 2020)
```

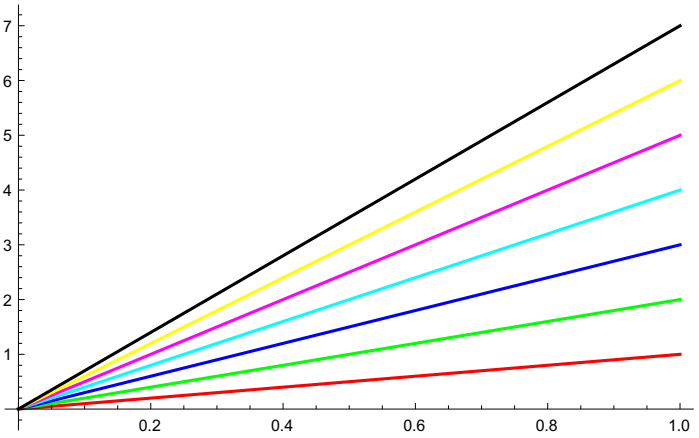
Setup:

In[2]:=
```
(* <<"BarCharts` ";<<"Histograms` ";<<"PieCharts` " *)
```

In[3]:=
```
colors = {Red, Green, Blue, Cyan, Magenta, Yellow, Black};
Protect[colors];
```

In[5]:=
```
Plot[Table[i x, {i, 1, 7}] // Evaluate, {x, 0, 1}, PlotStyle → colors]
```

Out[5]=



Example 1:

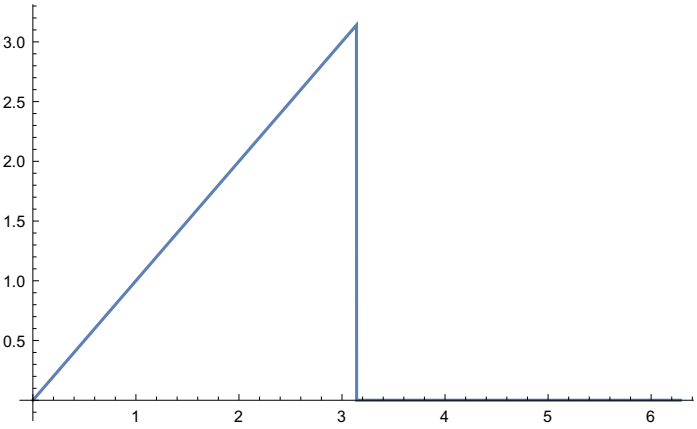
In[6]:=
```
Clear["Global`*"]
```
••• Clear : Symbol colors is Protected .

Plot the sample function:

In[7]:=
```
Clear[f];
f[x_]:=x /; x <= N[Pi];
f[x_]:=0 /; x >= N[Pi];
```

In[10]:=
```
Plot[f[x],{x,0,2  Pi}]
```

Out[10]=



Compute the Fourier coefficients

In[11]:=
```
c[0]= 1/(2  Pi) Integrate[  x   ,{x,0,1  Pi}]
```

Out[11]=
$$\frac{\pi}{4}$$

In[12]:=
```
c[m_]= 1/(2  Pi) Integrate[  x Exp[-  I  m  x] ,{x,0,1  Pi}] //FullSimplify
```

Out[12]=
$$\frac{-1+e^{-i\,m\,\pi}\,(1+i\,m\,\pi)}{2\,m^2\,\pi}$$

In[13]:=
```
term[0]= c[0];
```

In[14]:=
```
term[m_]= c[m] Exp[I  m  x] + c[-m]   Exp[-I  m  x]   //Simplify
```

Out[14]=
$$\frac{e^{-i\,m\,x}\left(-1+e^{i\,m\,\pi}\,(1-i\,m\,\pi)+e^{2\,i\,m\,x}\left(-1+e^{-i\,m\,\pi}\,(1+i\,m\,\pi)\right)\right)}{2\,m^2\,\pi}$$

In[15]:=
```
series[n_]:=  Sum[ term[m] ,{m,0,n}]
```

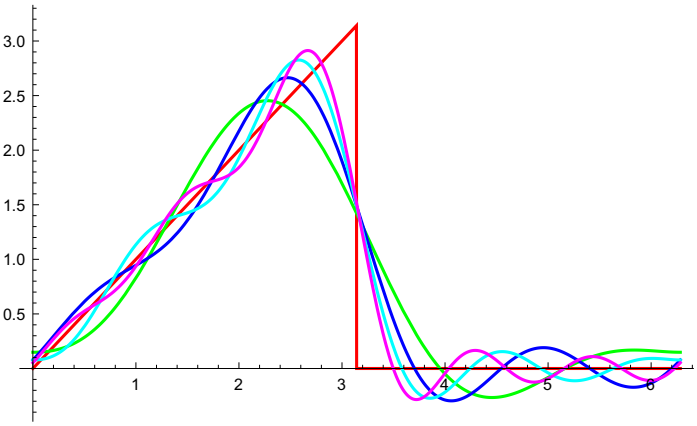
In[16]:=
```
series[2]
```

Out[16]=
$$\frac{e^{-i\,x}\left(-2+e^{2\,i\,x}\,(-2-i\,\pi)+i\,\pi\right)}{2\,\pi}+\frac{\pi}{4}+\frac{e^{-2\,i\,x}\left(-2\,i\,\pi+2\,i\,e^{4\,i\,x}\,\pi\right)}{8\,\pi}$$

Plot the series with different numbers of terms

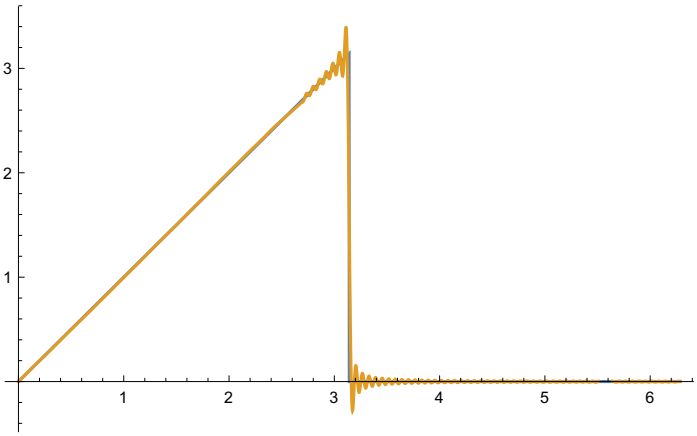
In[17]:=
```
Plot[ Join[{f[x]},Table[series[i],{i,2,5,1}]  ] //Evaluate
  ,{x,0,2  Pi},PlotStyle->colors]
```

Out[17]=



In[18]:=
```
Plot[ {f[x], series[100] } //Evaluate  ,{x,0,2  Pi}]
```

Out[18]=



Make a "frequency domain" plot of the coefficients

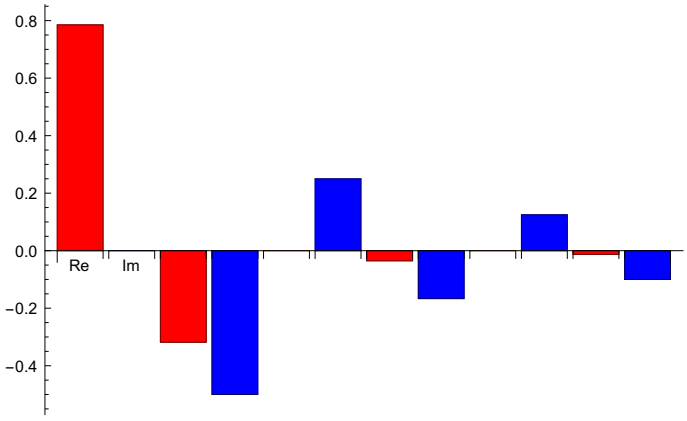
In[19]:=
```
coeffs = Table[{Re[c[i]], Im[c[i]]}, {i, 0, 5}] // Flatten
```

Out[19]=

$$\left\{\frac{\pi}{4}, 0, -\frac{1}{\pi}, -\frac{1}{2}, 0, \frac{1}{4}, -\frac{1}{9\,\pi}, -\frac{1}{6}, 0, \frac{1}{8}, -\frac{1}{25\,\pi}, -\frac{1}{10}\right\}$$

In[20]:=
```
BarChart[coeffs, ChartStyle → {Red, Blue}, ChartLabels → {"Re", "Im"}]
```

Out[20]=



Re-write the Exp series as a Trig series

In[21]:=
```
term2[0] = term[0] //ExpToTrig //Simplify
```

Out[21]=
$$\frac{\pi}{4}$$

In[22]:=
```
term2[m_]= term[m] //ExpToTrig //FullSimplify
```

Out[22]=
$$\frac{\text{Cos}[m\,(\pi - x)] - \text{Cos}[m\,x] + m\,\pi\,\text{Sin}[m\,(\pi - x)]}{m^2\,\pi}$$

In[23]:=
```
series2[n_]:= Sum[ term2[m] ,{m,0,n}]
```

In[24]:=
```
series[5]
```

Out[24]=
$$\frac{e^{-i\,x}\left(-2 + e^{2\,i\,x}\,(-2 - i\,\pi) + i\,\pi\right)}{2\,\pi} + \frac{e^{-3\,i\,x}\left(-2 + e^{6\,i\,x}\,(-2 - 3\,i\,\pi) + 3\,i\,\pi\right)}{18\,\pi} +$$
$$\frac{e^{-5\,i\,x}\left(-2 + e^{10\,i\,x}\,(-2 - 5\,i\,\pi) + 5\,i\,\pi\right)}{50\,\pi} + \frac{\pi}{4} + \frac{e^{-2\,i\,x}\left(-2\,i\,\pi + 2\,i\,e^{4\,i\,x}\,\pi\right)}{8\,\pi} + \frac{e^{-4\,i\,x}\left(-4\,i\,\pi + 4\,i\,e^{8\,i\,x}\,\pi\right)}{32\,\pi}$$

In[25]:=
```
series2[5]
```

Out[25]=
$$\frac{\pi}{4} + \frac{\text{Cos}[2\,(\pi - x)] - \text{Cos}[2\,x] + 2\,\pi\,\text{Sin}[2\,(\pi - x)]}{4\,\pi} +$$
$$\frac{\text{Cos}[3\,(\pi - x)] - \text{Cos}[3\,x] + 3\,\pi\,\text{Sin}[3\,(\pi - x)]}{9\,\pi} + \frac{\text{Cos}[4\,(\pi - x)] - \text{Cos}[4\,x] + 4\,\pi\,\text{Sin}[4\,(\pi - x)]}{16\,\pi} +$$
$$\frac{\text{Cos}[5\,(\pi - x)] - \text{Cos}[5\,x] + 5\,\pi\,\text{Sin}[5\,(\pi - x)]}{25\,\pi} + \frac{-2\,\text{Cos}[x] + \pi\,\text{Sin}[x]}{\pi}$$

In[26]:=
```
series2[5] // TrigReduce
```

Out[26]=
$$\frac{1}{900\,\pi}\left(225\,\pi^2 - 1800\,\text{Cos}[x] - 200\,\text{Cos}[3\,x] - 72\,\text{Cos}[5\,x] + \right.$$
$$\left. 900\,\pi\,\text{Sin}[x] - 450\,\pi\,\text{Sin}[2\,x] + 300\,\pi\,\text{Sin}[3\,x] - 225\,\pi\,\text{Sin}[4\,x] + 180\,\pi\,\text{Sin}[5\,x]\right)$$

Verify that they are identical

In[27]:=
```
series2[5] - series[5] // FullSimplify
```

Out[27]=
```
0
```

Problem 4: f[x_]=x for x=[0,2 π]

In[120]:=
```
Clear["Global`*"]
```

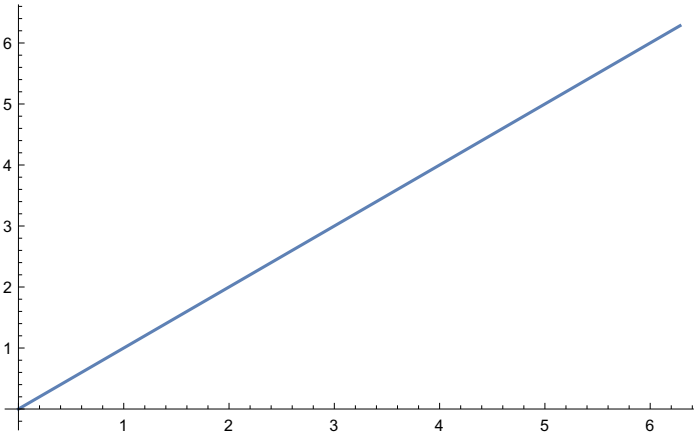
Clear : Symbol colors is Protected .

Part a)

In[121]:=
```
Clear[f];
f[x_]:=x;
```

In[123]:=
```
Plot[f[x],{x,0,2  Pi}]
```

Out[123]=



Compute the Fourier coefficients

In[124]:=
```
c[0]= 1/(2  Pi) Integrate[  f[x]   ,{x,0,2  Pi}]
```

Out[124]=
$$\pi$$

In[125]:=
```
c[m_]= 1/(2  Pi) Integrate[  f[x] Exp[- I  m  x] ,{x,0,2Pi}]  //FullSimplify
```

Out[125]=
$$\frac{-1+e^{-2\,i\,m\,\pi}\,(1+2\,i\,m\,\pi)}{2\,m^2\,\pi}$$

In[126]:=
```
term[0]= c[0];
```

In[127]:=
```
term[m_]= c[m] Exp[I  m  x] + c[-m]   Exp[-I  m  x]   //Simplify
```

Out[127]=
$$\frac{e^{-i\,m\,x}\left(-1+e^{2\,i\,m\,\pi}\,(1-2\,i\,m\,\pi)+e^{2\,i\,m\,x}\left(-1+e^{-2\,i\,m\,\pi}\,(1+2\,i\,m\,\pi)\right)\right)}{2\,m^2\,\pi}$$

In[128]:=
```
series[n_]:= Sum[ term[m] ,{m,0,n}]
```

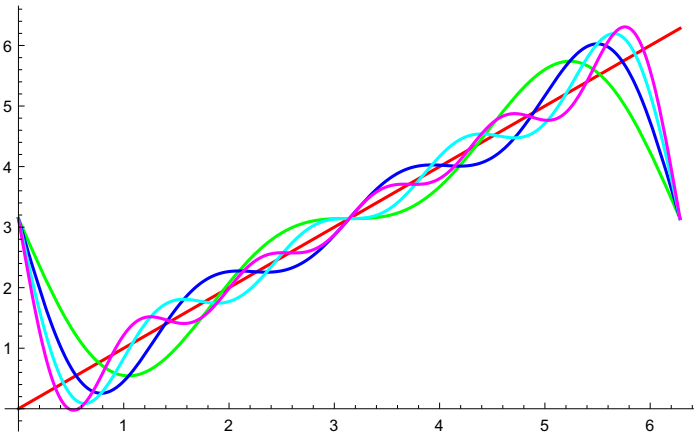
In[129]:=
```
series[2] // Simplify
```

Out[129]=
$$-i\, e^{-i\, x} + i\, e^{i\, x} - \frac{1}{2}\, i\, e^{-2\, i\, x} + \frac{1}{2}\, i\, e^{2\, i\, x} + \pi$$

Plot the series with different numbers of terms

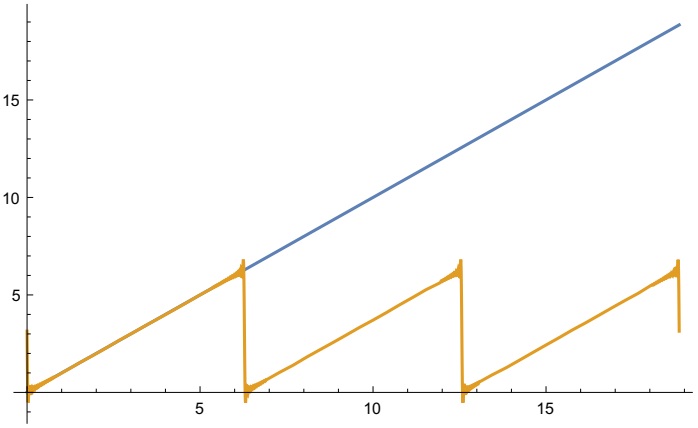
In[130]:=
```
Plot[ Join[{f[x]},Table[series[i],{i,2,5,1}]  ] //Evaluate
  ,{x,0,2  Pi},PlotStyle->colors]
```

Out[130]=



In[131]:=
```
Plot[ {f[x], series[100] } //Evaluate  ,{x,0,6  Pi}]
```

Out[131]=



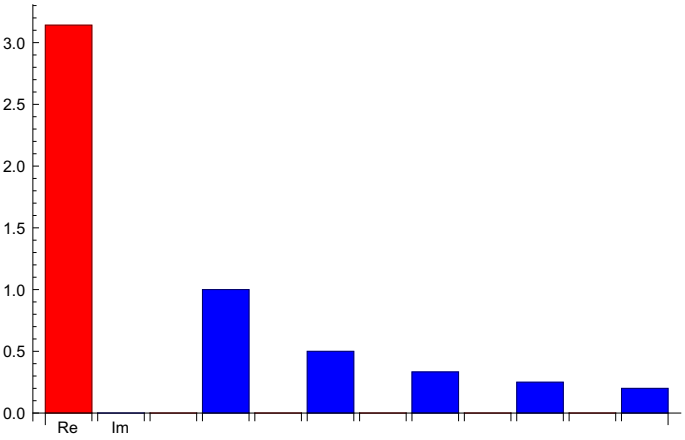
Make a "frequency domain" plot of the coefficients

In[132]:=
```
coeffs = Table[{Re[c[i]], Im[c[i]]}, {i, 0, 5}] // Flatten
```

Out[132]=
$$\left\{\pi,\ 0,\ 0,\ 1,\ 0,\ \frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{5}\right\}$$

In[133]:=
```
BarChart[coeffs, ChartStyle → {Red, Blue}, ChartLabels → {"Re", "Im"}]
```

Out[133]=



Re-write the Exp series as a Trig series

In[134]:=
```
term2[0] = term[0] //ExpToTrig //Simplify
```

Out[134]=
$$\pi$$

In[135]:=
```
term2[m_]= term[m] //ExpToTrig //FullSimplify
```

Out[135]=
$$\frac{\text{Cos}[m\,(2\,\pi - x)] - \text{Cos}[m\,x] + 2\,m\,\pi\,\text{Sin}[m\,(2\,\pi - x)]}{m^2\,\pi}$$

In[136]:=
```
series2[n_]:= Sum[ term2[m] ,{m,0,n}]
```

In[141]:=
```
series[5] //Expand
```

Out[141]=
$$-i\,e^{-i\,x} + i\,e^{i\,x} - \frac{1}{2}\,i\,e^{-2\,i\,x} + \frac{1}{2}\,i\,e^{2\,i\,x} - \frac{1}{3}\,i\,e^{-3\,i\,x} + \frac{1}{3}\,i\,e^{3\,i\,x} - \frac{1}{4}\,i\,e^{-4\,i\,x} + \frac{1}{4}\,i\,e^{4\,i\,x} - \frac{1}{5}\,i\,e^{-5\,i\,x} + \frac{1}{5}\,i\,e^{5\,i\,x} + \pi$$

In[138]:=
```
series2[5]
```

Out[138]=
$$\pi + \frac{\text{Cos}[2\,(2\,\pi - x)] - \text{Cos}[2\,x] + 4\,\pi\,\text{Sin}[2\,(2\,\pi - x)]}{4\,\pi} + \frac{\text{Cos}[3\,(2\,\pi - x)] - \text{Cos}[3\,x] + 6\,\pi\,\text{Sin}[3\,(2\,\pi - x)]}{9\,\pi} + \frac{\text{Cos}[4\,(2\,\pi - x)] - \text{Cos}[4\,x] + 8\,\pi\,\text{Sin}[4\,(2\,\pi - x)]}{16\,\pi} + \frac{\text{Cos}[5\,(2\,\pi - x)] - \text{Cos}[5\,x] + 10\,\pi\,\text{Sin}[5\,(2\,\pi - x)]}{25\,\pi} - 2\,\text{Sin}[x]$$

In[142]:=
```
series2[5] // TrigReduce // Expand
```

Out[142]=
$$\pi - 2\,\text{Sin}[x] - \text{Sin}[2\,x] - \frac{2}{3}\,\text{Sin}[3\,x] - \frac{1}{2}\,\text{Sin}[4\,x] - \frac{2}{5}\,\text{Sin}[5\,x]$$

Verify that they are identical

In[140]:=
```
series2[5] - series[5] // FullSimplify
```

Out[140]=
$$0$$

Problem 5: f[x_]=... for x=[0,2 π]

In[93]:=
```
Clear["Global`*"]
```

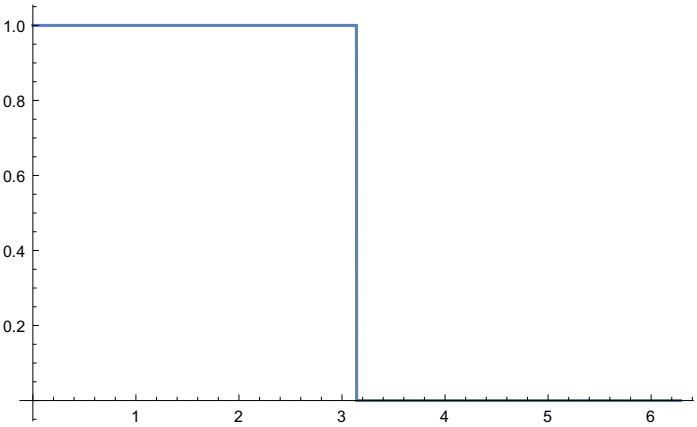
Clear : Symbol colors is Protected .

Part a)

In[94]:=
```
Clear[f];
f[x_]:=1 /; x <= N[Pi];
f[x_]:=0 /; x >= N[Pi];
```

In[97]:=
```
Plot[f[x],{x,0,2  Pi}]
```

Out[97]=



Compute the Fourier coefficients

In[98]:=
```
c[0]= 1/(2  Pi) Integrate[  1   ,{x,0,1  Pi}]
```

Out[98]=
$$\frac{1}{2}$$

In[99]:=
```
c[m_]= 1/(2  Pi) Integrate[  1 Exp[- I  m  x] ,{x,0,1  Pi}] //FullSimplify
```

Out[99]=
$$-\frac{i\left(1-e^{-i\,m\,\pi}\right)}{2\,m\,\pi}$$

In[100]:=
```
term[0]= c[0];
```

In[101]:=
```
term[m_]= c[m] Exp[I  m  x] + c[-m]  Exp[-I  m  x]  //Simplify
```

Out[101]=
$$-\frac{i\,e^{-i\,m\,(\pi+x)}\left(-1+e^{i\,m\,\pi}\right)\left(e^{i\,m\,\pi}+e^{2\,i\,m\,x}\right)}{2\,m\,\pi}$$

In[102]:=
```
series[n_]:=  Sum[ term[m] ,{m,0,n}]
```

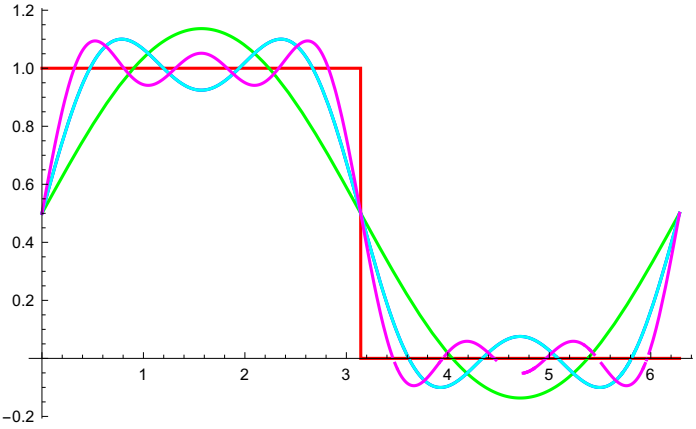
In[103]:=
```
series[2]
```

Out[103]=
$$\frac{1}{2}+\frac{i\,e^{-i\,(\pi+x)}\left(-1+e^{2\,i\,x}\right)}{\pi}$$

Plot the series with different numbers of terms

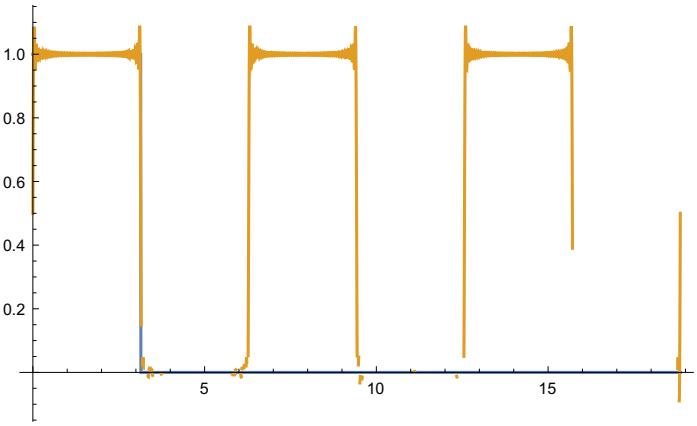
In[104]:=
```
Plot[ Join[{f[x]},Table[series[i],{i,2,5,1}]  ] //Evaluate
  ,{x,0,2  Pi},PlotStyle->colors]
```

Out[104]=



In[105]:=
```
Plot[ {f[x], series[100] } //Evaluate  ,{x,0,6  Pi}]
```

Out[105]=



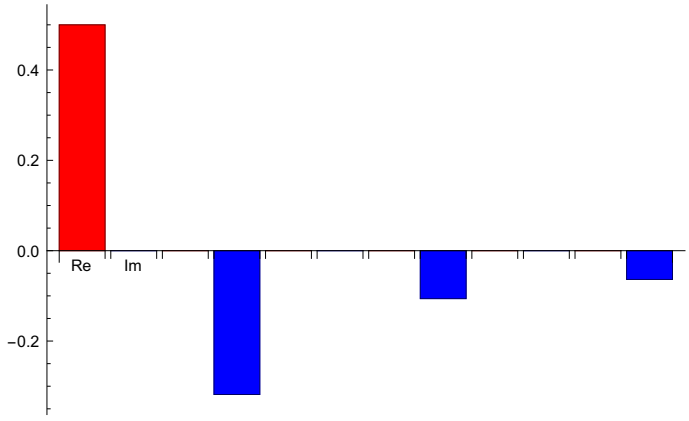
Make a "frequency domain" plot of the coefficients

In[106]:=
```
coeffs = Table[{Re[c[i]], Im[c[i]]}, {i, 0, 5}] // Flatten
```

Out[106]=
$$\left\{\frac{1}{2},\ 0,\ 0,\ -\frac{1}{\pi},\ 0,\ 0,\ 0,\ -\frac{1}{3\,\pi},\ 0,\ 0,\ 0,\ -\frac{1}{5\,\pi}\right\}$$

In[107]:=
```
BarChart[coeffs, ChartStyle → {Red, Blue}, ChartLabels → {"Re", "Im"}]
```

Out[107]=



Re-write the Exp series as a Trig series

In[108]:=
```
term2[0] = term[0] //ExpToTrig //Simplify
```

Out[108]=
$$\frac{1}{2}$$

In[109]:=
```
term2[m_]= term[m] //ExpToTrig //FullSimplify
```

Out[109]=
$$\frac{\text{Sin}[m\,(\pi - x)] + \text{Sin}[m\,x]}{m\,\pi}$$

In[110]:=
```
series2[n_]:= Sum[ term2[m] ,{m,0,n}]
```

In[116]:=
```
series[5] //ExpandAll
```

Out[116]=
$$\frac{1}{2} + \frac{i\,e^{-i\,x}}{\pi} - \frac{i\,e^{i\,x}}{\pi} + \frac{i\,e^{-3\,i\,x}}{3\,\pi} - \frac{i\,e^{3\,i\,x}}{3\,\pi} + \frac{i\,e^{-5\,i\,x}}{5\,\pi} - \frac{i\,e^{5\,i\,x}}{5\,\pi}$$

In[117]:=
```
series2[5]
```

Out[117]=
$$\frac{1}{2} + \frac{2\,\text{Sin}[x]}{\pi} + \frac{\text{Sin}[2\,(\pi - x)] + \text{Sin}[2\,x]}{2\,\pi} + \frac{\text{Sin}[3\,(\pi - x)] + \text{Sin}[3\,x]}{3\,\pi} + \frac{\text{Sin}[4\,(\pi - x)] + \text{Sin}[4\,x]}{4\,\pi} + \frac{\text{Sin}[5\,(\pi - x)] + \text{Sin}[5\,x]}{5\,\pi}$$

In[119]:=
```
series2[5] // TrigReduce // Expand
```

Out[119]=
$$\frac{1}{2} + \frac{2\,\text{Sin}[x]}{\pi} + \frac{2\,\text{Sin}[3\,x]}{3\,\pi} + \frac{2\,\text{Sin}[5\,x]}{5\,\pi}$$

Verify that they are identical

In[114]:=
```
series2[5] - series[5] // FullSimplify
```

Out[114]=
```
0
```

Problem 6: f[x_]=... for x=[0,2 π]

In[242]:=
```
Clear["Global`*"]
```

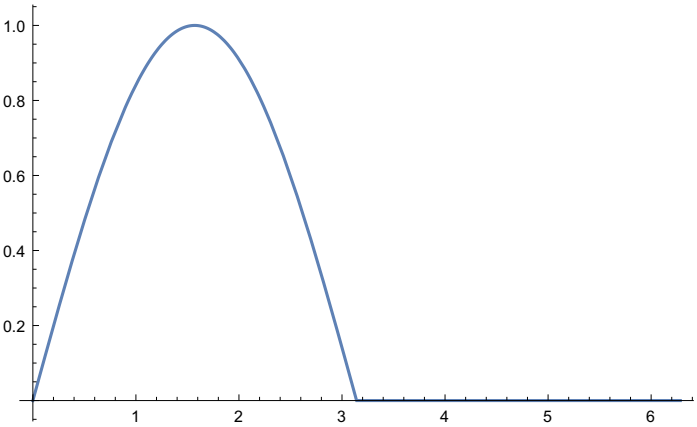
Clear : Symbol colors is Protected .

Part a)

In[243]:=
```
Clear[f];
f[x_]:=Sin[x] /; x <= N[Pi];
f[x_]:=0 /; x >= N[Pi];
```

In[246]:=
```
Plot[f[x],{x,0,2 Pi}]
```

Out[246]=



Compute the Fourier coefficients

In[247]:=
```
c[0]= 1/(2 Pi) Integrate[ Sin[x]  ,{x,0,1 Pi}]
```

Out[247]=
$$\frac{1}{\pi}$$

In[248]:=
```
c[m_]= 1/(2 Pi) Integrate[ Sin[x] Exp[- I m x] ,{x,0,1 Pi}] //FullSimplify
```

Out[248]=
$$\frac{1+e^{-i\,m\,\pi}}{2\,\pi-2\,m^2\,\pi}$$

Treat c {+1,-1 } terms specially

In[249]:=
```
{c[+1], c[-1]}
```

Power : Infinite expression $\frac{1}{0}$ encountered .

Infinity : Indeterminate expression 0 ComplexInfinity encountered .

Power : Infinite expression $\frac{1}{0}$ encountered .

Infinity : Indeterminate expression 0 ComplexInfinity encountered .

Out[249]=
```
{Indeterminate , Indeterminate }
```

In[250]:=
```
{c[+1], c[-1]} = Limit[c[m], m → {+1, -1}]
```

Out[250]= $\left\{-\frac{i}{4}, \frac{i}{4}\right\}$

In[251]:=
```
term[0]= c[0];
```

In[252]:=
```
term[m_]= c[m] Exp[I  m  x] + c[-m]  Exp[-I  m  x]  //Simplify
```

Out[252]= $-\frac{e^{-i\,m\,(\pi+x)}\left(1+e^{i\,m\,\pi}\right)\left(e^{i\,m\,\pi}+e^{2\,i\,m\,x}\right)}{2\left(-1+m^2\right)\pi}$

Treat {+1,-1 } terms specially

In[253]:=
```
{term[1], term[-1]}
```

Power : Infinite expression $\frac{1}{0}$ encountered .

Infinity : Indeterminate expression $\frac{0\,e^{-i\,(\pi+x)}\left(-1+e^{2\,i\,x}\right)\text{ComplexInfinity}}{\pi}$ encountered .

Power : Infinite expression $\frac{1}{0}$ encountered .

Infinity : Indeterminate expression $\frac{0\,e^{i\,(\pi+x)}\left(-1+e^{-2\,i\,x}\right)\text{ComplexInfinity}}{\pi}$ encountered .

Out[253]=
```
{Indeterminate , Indeterminate }
```

In[254]:=
```
tmp = Limit[term[m], m -> {+1, -1}] // Expand
```

Out[254]= $\left\{\frac{1}{4}\,i\,e^{-i\,x}-\frac{1}{4}\,i\,e^{i\,x},\ \frac{1}{4}\,i\,e^{-i\,x}-\frac{1}{4}\,i\,e^{i\,x}\right\}$

In[255]:=
```
{term[1], term[-1]} = tmp
```

Out[255]= $\left\{\frac{1}{4}\,i\,e^{-i\,x}-\frac{1}{4}\,i\,e^{i\,x},\ \frac{1}{4}\,i\,e^{-i\,x}-\frac{1}{4}\,i\,e^{i\,x}\right\}$

In[256]:=
```
series[n_]:=  Sum[ term[m] ,{m,0,n}]
```

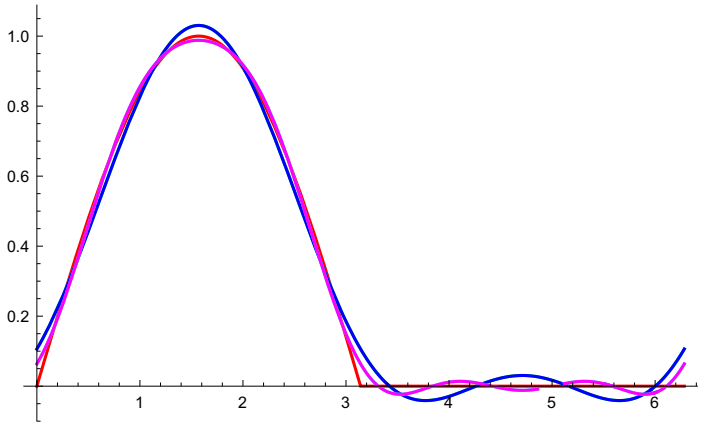
In[257]:=
```
series[4] // ExpandAll
```

Out[257]=
$$\frac{1}{4} i e^{-i x} - \frac{1}{4} i e^{i x} + \frac{1}{\pi} - \frac{e^{-2 i x}}{3 \pi} - \frac{e^{2 i x}}{3 \pi} - \frac{e^{-4 i x}}{15 \pi} - \frac{e^{4 i x}}{15 \pi}$$

Plot the series with different numbers of terms

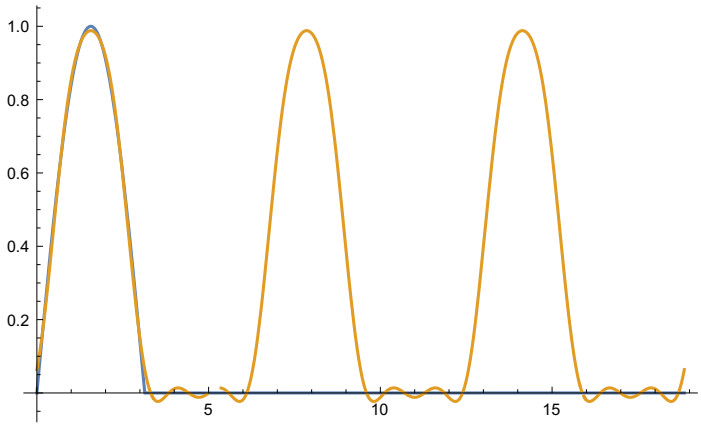
In[258]:=
```
Plot[ Join[{f[x]},Table[series[i],{i,2,5,1}]  ] //Evaluate
  ,{x,0,2  Pi},PlotStyle->colors]
```

Out[258]=



In[259]:=
```
Plot[ {f[x], series[5] } //Evaluate  ,{x,0,6  Pi}]
```

Out[259]=



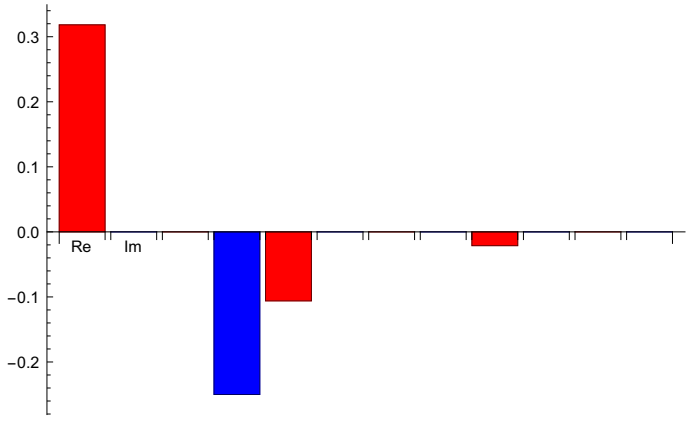
Make a "frequency domain" plot of the coefficients

In[260]:=
```
coeffs = Table[{Re[c[i]], Im[c[i]]}, {i, 0, 5}] // Flatten
```

Out[260]=
$$\left\{\frac{1}{\pi}, 0, 0, -\frac{1}{4}, -\frac{1}{3 \pi}, 0, 0, 0, -\frac{1}{15 \pi}, 0, 0, 0\right\}$$

In[261]:=
```
BarChart[coeffs, ChartStyle → {Red, Blue}, ChartLabels → {"Re", "Im"}]
```

Out[261]=



Re-write the Exp series as a Trig series

In[262]:=
```
term2[0] = term[0] //ExpToTrig //Simplify
```

Out[262]=
$$\frac{1}{\pi}$$

In[270]:=
```
term2[1] = term[1] // ExpToTrig // Simplify
```

Out[270]=
$$\frac{\text{Sin}[x]}{2}$$

In[271]:=
```
term2[m_]= term[m] //ExpToTrig //FullSimplify
```

Out[271]=
$$\frac{\text{Cos}[m\,(\pi - x)] + \text{Cos}[m\,x]}{\pi - m^2\,\pi}$$

In[272]:=
```
series2[n_]:= Sum[ term2[m] ,{m,0,n}]
```

In[275]:=
```
series[5] //ExpandAll
```

Out[275]=
$$\frac{1}{4}\,i\,e^{-i\,x} - \frac{1}{4}\,i\,e^{i\,x} + \frac{1}{\pi} - \frac{e^{-2\,i\,x}}{3\,\pi} - \frac{e^{2\,i\,x}}{3\,\pi} - \frac{e^{-4\,i\,x}}{15\,\pi} - \frac{e^{4\,i\,x}}{15\,\pi}$$

In[276]:=
```
series2[5] //ExpandAll
```

Out[276]=
$$\frac{1}{\pi} - \frac{2\,\text{Cos}[2\,x]}{3\,\pi} - \frac{2\,\text{Cos}[4\,x]}{15\,\pi} + \frac{\text{Sin}[x]}{2}$$

In[278]:=
```
series2[5] // TrigReduce // ExpandAll
```

Out[278]=
$$\frac{1}{\pi} - \frac{2\,\text{Cos}[2\,x]}{3\,\pi} - \frac{2\,\text{Cos}[4\,x]}{15\,\pi} + \frac{\text{Sin}[x]}{2}$$

Verify that they are identical

In[279]:=
```
series2[5] - series[5] // FullSimplify
```

Out[279]=
```
0
```