Initialization for Lagrangian Problems

Problem 3: Bead on a Rotating Hoop

check DBC for correct code

Remarks and outline

Solution

In[ ]:=
```
Clear["Global`*"]
```

Part a

In[ ]:=
```
x2rRule = {x, y, z} → {(r Sin[θ[#]] Cos[ω #]) &, (r Sin[θ[#]] Sin[ω #]) &, (r Cos[θ[#]]) &} // Thread;
```

In[ ]:=
```
ψRule = {θ → (π - ψ[#] &)};
```

In[ ]:= $T = \frac{1}{2} m \left(x'[t]^2 + y'[t]^2 + z'[t]^2\right)$ `//. x2rRule /. ψRule // Simplify`

Out[ ]= $\frac{1}{2} m r^2 \left(\omega^2 \text{Sin}[\psi[t]]^2 + \psi'[t]^2\right)$

In[ ]:=
```
V = m g z[t] //. x2rRule /. ψRule // Simplify
```

Out[ ]= $-g\, m\, r\, \text{Cos}[\psi[t]]$

In[ ]:=
```
L = (T - V)
```

Out[ ]= $g\, m\, r\, \text{Cos}[\psi[t]] + \frac{1}{2} m r^2 \left(\omega^2 \text{Sin}[\psi[t]]^2 + \psi'[t]^2\right)$

In[ ]:=
```
eq1 = EulerEquations[L, ψ[t], t]
```

Out[ ]= $m\, r \left(\left(-g + r\, \omega^2 \text{Cos}[\psi[t]]\right) \text{Sin}[\psi[t]] - r\, \psi''[t]\right) == 0$

In[ ]:= `ψSol = (Solve[eq1, ψ''[t]] // Flatten) /.` $\{g \to \omega c^2\, r\}$ `// ExpandAll`

Out[ ]= $\{\psi''[t] \to -\omega c^2 \text{Sin}[\psi[t]] + \omega^2 \text{Cos}[\psi[t]] \text{Sin}[\psi[t]]\}$

In[ ]:= `energy = (ψ'[t] × D[L, ψ'[t]] - L) /.` $\{g \to \omega c^2\, r\}$ `// Simplify`

Out[ ]= $-\frac{1}{2} m r^2 \left(2\, \omega c^2 \text{Cos}[\psi[t]] + \omega^2 \text{Sin}[\psi[t]]^2 - \psi'[t]^2\right)$

In[ ]:=
```
D[energy, t] /. ψSol /. {g → ωc^2 r} // Simplify
```

Out[ ]= 0

In[ ]:= `FirstIntegrals[L, ψ[t], t] /.` $\{g \to \omega c^2\, r\}$ `// Simplify`

Out[ ]= $\left\{\text{FirstIntegral}[t] \to -\frac{1}{2} m r^2 \left(2\, \omega c^2 \text{Cos}[\psi[t]] + \omega^2 \text{Sin}[\psi[t]]^2 - \psi'[t]^2\right)\right\}$

Part b

In[ ]:=
```
Veff = energy //. {ψ'[t] → 0} // ExpandAll
```

Out[ ]= $-m\, r^2\, \omega c^2 \,\text{Cos}[\psi[t]] - \frac{1}{2} m\, r^2\, \omega^2\, \text{Sin}[\psi[t]]^2$

In[ ]:=
```
dVeff= D[Veff,ψ[t]]
```

Out[ ]= $m\, r^2\, \omega c^2\, \text{Sin}[\psi[t]] - m\, r^2\, \omega^2\, \text{Cos}[\psi[t]]\, \text{Sin}[\psi[t]]$

In[ ]:=
```
(*sol=Solve[dVeff ==0,ψ[t],InverseFunctions →True ] *)
sol = Solve[dVeff == 0, ψ[t], InverseFunctions → True ] /. C[1]→ 0 // Simplify
```

Out[ ]= $\left\{\{\psi[t]\to 0\},\ \{\psi[t]\to \pi\},\ \left\{\psi[t]\to \text{ArcTan}\left[\frac{\omega c^2}{\omega^2},\ -\frac{\sqrt{\omega^4-\omega c^4}}{\omega^2}\right]\right\},\ \left\{\psi[t]\to \text{ArcTan}\left[\frac{\omega c^2}{\omega^2},\ \frac{\sqrt{\omega^4-\omega c^4}}{\omega^2}\right]\right\}\right\}$

In[ ]:=
(*$\left\{\{\psi[t]\to 0\},\ \left\{\psi[t]\to -\text{ArcCos}\left[\frac{\omega c^2}{\omega^2}\right]\right\},\ \left\{\psi[t]\to \text{ArcCos}\left[\frac{\omega c^2}{\omega^2}\right]\right\}\right\}$ -should output^^^^^^^*)

testRule = $-\sqrt{\omega^2} \le \omega c \le \sqrt{\omega^2}$

```
sol = Simplify[Solve[dVeff == 0, ψ[t], Reals , InverseFunctions → True] /. C[1]→ 0,
   Assumptions → testRule]
(*eqRule=Join[{{ψ[t]→π}},sol] *)(*THIS LINE IS OBSOLETE
 BECAUSE THE FIX GIVES ALL THE EQUILIBRIUM POINTS*)
```

Out[ ]= $-\sqrt{\omega^2} \le \omega c \le \sqrt{\omega^2}$

Out[ ]= $\left\{\{\psi[t]\to 0\},\ \{\psi[t]\to \pi\},\ \left\{\psi[t]\to -\text{ArcCos}\left[\frac{\omega c^2}{\omega^2}\right]\right\},\ \left\{\psi[t]\to \text{ArcCos}\left[\frac{\omega c^2}{\omega^2}\right]\right\}\right\}$

In[ ]:=
```
dVeff == 0 //. sol
```

Out[ ]= {True , True , True , True}

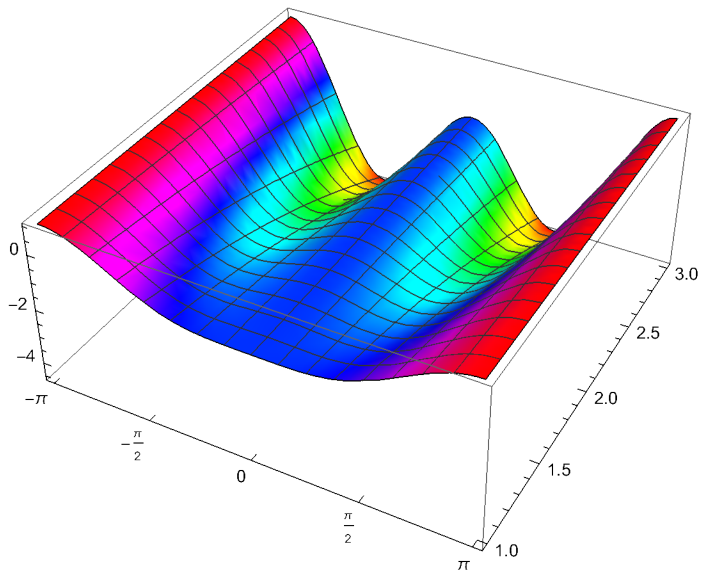
In[ ]:=

```
(*{True , True , True , True} -should output^^^^^^^*)
```

In[ ]:=
```
values1 = {r → 1, ωc → 1, m → 1};
```

In[ ]:=
```
Plot3D[Evaluate[Veff /.values1], {ψ[t], -π, π}, {ω, 1, 3},
  ColorFunction → Hue, Ticks → {π Range[-1, 1, 1/2], Automatic, Automatic}]
```

Out[ ]=



In[ ]:=
```
Plot[Evaluate[(Veff //.values1 /.ω → #1 &) /@ {2, 0.2`}],
  {ψ[t], -π, π}, Ticks → {{-π, 0, π}, Automatic},
  Epilog → {Hue[0.6`], AbsolutePointSize[6], Point /@ ({ψ[t], Veff} /.sol /.values1 /.ω → 2)}]
(*REPLACED ALL eqRule→sol*)
```

Out[ ]=