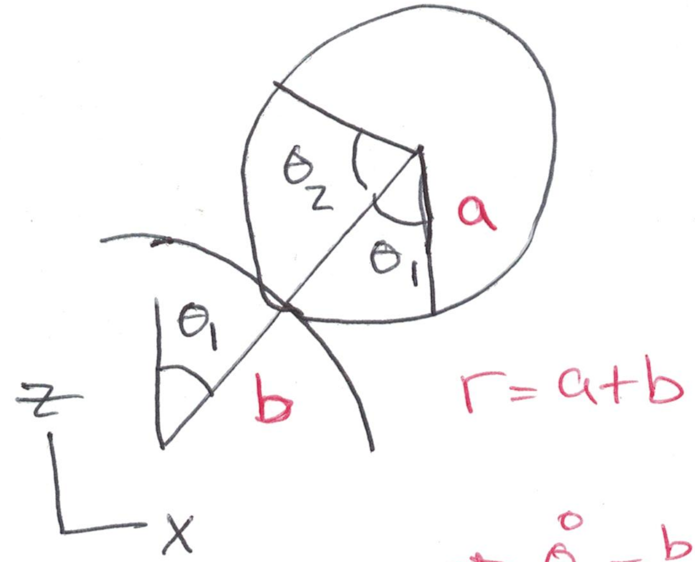
Homework #6

Constraint Equations

① $r = a + b$

② $b\theta_1 - a\theta_2 = 0$

use ② to eliminate θ_2



$$\theta_2 = \frac{b}{a}\theta_1$$

$$\dot{\theta}_2 = \frac{b}{a}\dot{\theta}_1$$

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}_1^2\right] + \frac{I}{2}\left[(\dot{\theta}_1 + \dot{\theta}_2)^2\right]$$

Linear T   Rotational T

$$V = mgh = mgr\cos\theta_1$$

Equation of Constraint: $F \equiv r - a - b = 0$

$$\frac{\partial F}{\partial r} = 1 \qquad \frac{\partial F}{\partial \theta} = 0$$

Variables $\{r, \dot{r}, \theta_1, \dot{\theta}_1\}$ we have eliminated θ_2

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \lambda\frac{\partial F}{\partial r}$$

$\boxed{\text{Hint}}$ $\dot{r} = \ddot{r} = 0$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \lambda\frac{\partial F}{\partial \theta_1}$$

Req: $m g \cos\theta_1 - m r \dot{\theta}_1^2 + \cancelto{0}{m \ddot{r}} = \lambda$

θeq: $-m g r \sin\theta_1 + \cancelto{0}{2 m r \dot{r}}\, \dot{\theta}_1 + m r^2 \ddot{\theta}_1$

$$+ \left(\frac{a+b}{a}\right)^2 I \ddot{\theta}_1 = 0$$

$r = a+b \Rightarrow \ddot{r} = 0 \qquad \dot{r} = 0$

$\boxed{\text{Solve Req for } \dot{\theta}_1}$

$$\dot{\theta}_1^2 = \frac{m g \cos\theta - \lambda}{m r}$$

$\boxed{\text{Solve } \theta\text{eq for } \ddot{\theta}_1}$ $\quad \dot{r} = \ddot{r} = 0 \qquad a+b = r$

θeq: $-m g r \sin\theta_1 + m r^2 \ddot{\theta}_1 + \frac{r^2}{a^2} I \ddot{\theta}_1 = 0$

$$\ddot{\theta}_1 = \frac{a^2 m g \sin\theta_1}{r(I + a^2 m)}$$

$$\ddot{\theta}_1 - \underbrace{\left[\frac{a^2 m g}{r(I + a^2 m)}\right]}_{z} \sin\theta_1 = 0$$

$$\int_{\dot{\theta}_1 = 0 \;\; \theta_1 = 0}^{\dot{\theta}_1 \;\; \theta_1} \left[\ddot{\theta}_1 - z \sin\theta_1\right] \dot{\theta} = \left. \frac{1}{2}\dot{\theta}_1^2 + z \cos\theta_1 \right]$$

Integration Limits

$$\theta_1 = [0, \theta_1] \quad \dot{\theta}_1 = [0, \dot{\theta}_1]$$

$$0 = \left[\frac{1}{2}\dot{\theta}_1 + z\cos\theta_1\right] - \left[0 + z\cos(0)\right]$$

$$0 = \frac{1}{2}\dot{\theta}_1 + z(\cos\theta_1 - 1)$$

$$\dot{\theta}_1 = 2z(1-\cos\theta_1) = \frac{2a^2mg}{r(I+a^2m)}(1-\cos\theta)$$

Substitute back in Req:

$$\text{Req}: \quad mg\cos\theta_1 - mr\dot{\theta}_1^2 = \lambda$$

$$\lambda = mg\cos\theta_1 - mr\frac{2a^2mg}{r(I+a^2m)}(1-\cos\theta_1)$$

Find when $\lambda = 0$:

$$0 = \cos\theta_1 - \underbrace{\left[\frac{2a^2m}{I+a^2m}\right]}_{W}(1-\cos\theta_1)$$

$$0 = C - W(1-C) = C - W + WC$$

$$\cos\theta = C = \frac{W}{1+W} =$$

$$W = \frac{2a^2 m}{I + a^2 m}$$

Let $I = K a^2 m$ where $\begin{cases} \text{hoop } K = 1 \\ \text{Sphere } K = 2/5 \\ \text{Disk } K = 1/2 \end{cases}$

$$W = \frac{2a^2 m}{Ka^2 m + a^2 m} = \frac{2}{K+1}$$

$$\cos\theta = \frac{W}{1+W} = \frac{2/K+1}{1 + \frac{2}{K+1}} = \frac{2/K+1}{\frac{K+1+2}{K+1}}$$

$$\cos\theta = \frac{2}{3+K}$$

$K = 1$ $\cos\theta = \frac{1}{2}$ $\theta \sim 60^\circ$

$K = 1/2$ $\cos\theta = \frac{4}{7}$ $\theta \sim 55.15^\circ$

$K = 2/5$ $\cos\theta = \frac{10}{17}$ $\theta \sim 53.97^\circ$