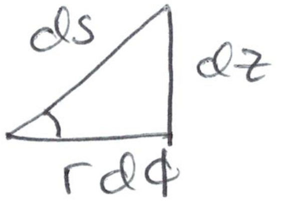
#1) Shortest distance on a cylinder

r is fixed



$$ds^2 = dz^2 + r^2 d\phi^2$$

$$F(z,\dot{z},\phi,\dot{\phi},u) = s = \sqrt{\dot{z}^2 + r^2\dot{\phi}^2}$$

F does not depend on $\{z, \phi\}$

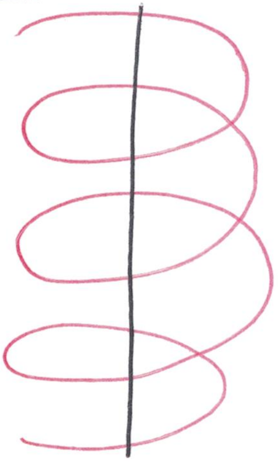
$$\therefore \quad \frac{\partial F}{\partial \dot{z}} = \frac{\dot{z}}{\sqrt{\dot{z}^2 + r^2\dot{\phi}^2}} = c_1 \quad = \frac{\partial F}{\partial \dot{\phi}} = \frac{r^2\dot{\phi}}{\sqrt{\dot{z}^2 + r^2\dot{\phi}^2}} = c_2$$

$$\Rightarrow \quad \dot{z} \propto r^2 \phi \quad \Rightarrow \quad \dot{\phi} = c\dot{z}$$

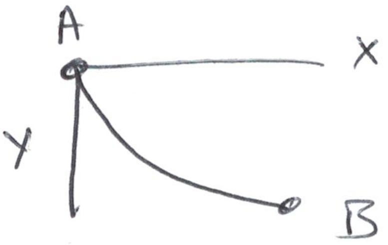
(constant

$$\int \dot{\phi} = \int c\,\dot{z} \quad \Rightarrow \quad \phi = cz + \phi_0$$

$\boxed{\text{Helix}}$



Brachistochrone Problem :



$X = vt \qquad t = \frac{X}{v}$

$v = \sqrt{2gy}$ since $\frac{1}{2}mv^2 = mgy$

$$\text{time} = \int_A^B dt = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}} = \int \frac{\sqrt{\frac{dx^2}{dy^2}+1}}{\sqrt{2gy}}\,dy$$

$$\text{time} = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{\dot{X}^2+1}}{\sqrt{Y}}\,dY = \frac{1}{\sqrt{2g}} \int F(X,\dot{X})\,dY$$

Now this is in the form of the Euler-Lagrange.

$$\frac{\partial F}{\partial X} - \frac{d}{dY}\frac{\partial F}{\partial \dot{X}} = 0 \qquad \text{but} \quad \frac{\partial F}{\partial X} = 0$$

$$\therefore \quad \frac{\partial F}{\partial \dot{X}} = \text{const} = \frac{1}{\sqrt{Y}}\frac{\dot{X}}{\sqrt{\dot{X}^2+1}}$$

Square this for convenience.

$$\frac{\dot{X}^2}{Y(1+\dot{X}^2)} = \text{const} = \frac{1}{2a} \qquad (\text{Eq } 6.22)$$

Solution: Brachistochrone

$$\Rightarrow \quad 2a\dot{X}^2 = Y + Y\dot{X}^2$$

$$\dot{X}^2 = \frac{Y}{2a-Y} = \left(\frac{dX}{dY}\right)^2$$

$$X = \int dY \sqrt{\frac{Y}{2a-Y}} = \int a\sin\theta \underbrace{\sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}}_{\text{Tan}\left[\frac{\theta}{2}\right]}$$

$$Y = a(1-\cos\theta)$$

$$dY = a\sin\theta$$

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\text{Tan}\left(\frac{\theta}{2}\right) = \frac{\sin(\theta/2)}{\cos(\theta/2)}$$

$$X = a\int 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{\cos(\theta/2)}$$

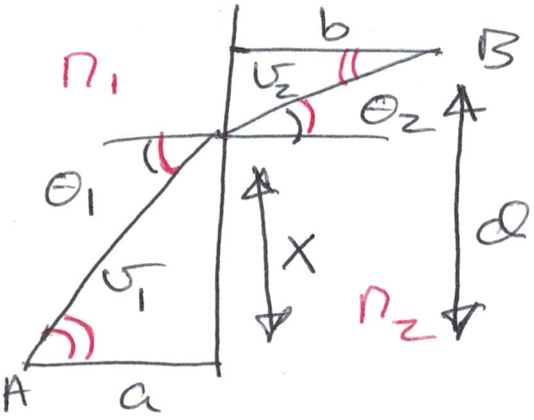
$$X = 2a\int \sin^2\left(\frac{\theta}{2}\right) = a\int 1-\cos(\theta)$$

$$X = a\int_0^\theta d\theta\,[1-\cos\theta] = a[\theta-\sin\theta]_0^\theta$$

$$X = a(\theta-\sin\theta) \qquad Y = a(1-\cos\theta)$$

Plot this

#3] Snells Law



$x = vt \qquad t = \frac{x}{v}$

$$t = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}$$

$$\frac{dt}{dx} = 0 = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{(d-x)}{v_2\sqrt{b^2+(d-x)^2}}$$

$$0 = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

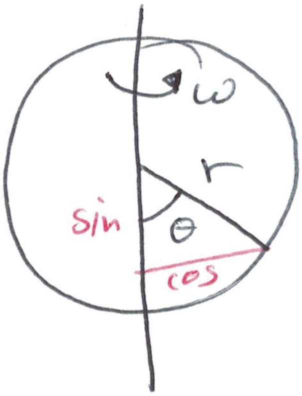
$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}}$$

Note Reversed!!!

Since $v = \frac{c}{n}$

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

Bead on Hoop



$$T = \frac{m}{2}\left(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\omega^2\right)$$

$$V = -mgr\cos\theta$$

$$L = T - V$$

a minus from $-V$ and from $d\cos\theta$

$$\frac{\partial L}{\partial \theta} = m r^2 \sin\theta\cos\theta\,\omega^2 - mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$-mr^2\ddot{\theta} + mr^2\sin\theta\cos\theta\,\omega^2 - mgr\sin\theta = 0$$

$$\ddot{\theta} - \sin\theta\cos\theta\,\omega^2 + \frac{g}{r}\sin\theta = 0$$

Solve for $\ddot{\theta} = 0$

$\boxed{\text{Eq}}$ $$\ddot{\theta} = \sin\theta\left[\cos\theta\,\omega^2 - \frac{g}{r}\right] = 0$$

$$\sin\theta = 0 \Rightarrow \boxed{\theta = 0, \pi}$$

$$\cos\theta = \frac{g/r}{\omega^2} \quad \text{if} \quad \frac{g/r}{\omega^2} \le 1$$

$$\omega_c^2 = g/r \Rightarrow \boxed{\omega_c^2 \le \omega^2}$$

Equilibrium Points

$\theta = 0, \pi$ $\qquad \cos\theta = \frac{\omega_c^2}{\omega^2}$ if $\omega_c < \omega$

For $\theta \cong 0$ small $\qquad \sin\theta \sim \theta \qquad \cos\theta \sim 1$

$$\boxed{Eq} \Rightarrow \ddot{\theta} - \theta\omega^2 + \overset{\omega_c^2}{\frac{g}{r}}\theta = 0$$

$$\ddot{\theta} + (\omega_c^2 - \omega^2)\theta = 0$$

$$\ddot{\theta} + \omega_{eff}^2\theta = 0$$

$$\omega_{eff} = \sqrt{\omega_c^2 - \omega^2} \qquad \theta = e^{\pm i\omega_{eff}t}$$

IF $\omega < \omega_c$, ω_{eff} is real and the solution oscilate

IF $\omega > \omega_c$ then $\theta = e^{\pm\omega_{eff}t}$ and <u>NO</u> oscillation

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For $\theta \sim \pi + \varepsilon \qquad \sin\theta \sim -\varepsilon \qquad \cos\theta \sim -1$

$$\ddot{\theta} - \theta\omega^2 - \omega_c^2\theta = 0$$

unstable

$$\ddot{\theta} - (\omega^2 + \omega_c^2)\theta = 0$$

$$\theta = e^{\pm\omega_{eff}t}$$

No Oscillation

$$\boxed{Eq} \qquad \ddot{\theta} - \sin\theta\,\omega^2\left[\cos\theta - \frac{\omega_c^2}{\omega^2}\right] = 0$$

where $\omega_c^2 = g/r$

When $\cos\theta_e = \dfrac{g/r}{\omega^2} = \dfrac{\omega_c^2}{\omega^2}$ $\therefore$ $\theta_e = \cos^{-1}\left(\dfrac{\omega_c^2}{\omega^2}\right)$

Let $\theta = \underset{\substack{\uparrow \\ \text{equilibrium}}}{\theta_e} + \underset{\nwarrow\ \text{small}}{\varepsilon}$ $\qquad \ddot{\theta} = \ddot{\varepsilon}$

$$\cos[\theta_c + \varepsilon] = \cos\theta_c \cos\varepsilon - \sin\theta_e \sin\varepsilon$$

$$\simeq \cos\theta_c \cdot 1 - \sin\theta_e\, \varepsilon$$

$$\sin[\theta_c + \varepsilon] = \sin\theta_c \cos\varepsilon + \cos\theta_c \sin\varepsilon$$

$$\simeq \sin\theta_c \cdot 1 + \cos\theta_c\, \varepsilon$$

$$\boxed{Eq} \Rightarrow \ddot{\varepsilon} - \omega^2\left[\sin\theta_e + \cos\theta_c\,\varepsilon\right] \times$$

$$\times\left[\cos\theta_e - \sin\theta_e\,\varepsilon - \frac{\omega_c^2}{\omega^2}\right]$$

$\leftrightarrow \dfrac{\omega_c^2}{\omega^2}$ $\quad$ cancel $\leftrightarrow$

$$\Rightarrow \ddot{\varepsilon} + \underbrace{\sin^2\theta_c\,\omega^2}_{\omega_{\text{eff}}^2}\,\varepsilon = 0 \leftarrow \text{drop } \varepsilon^2 \text{ terms}$$