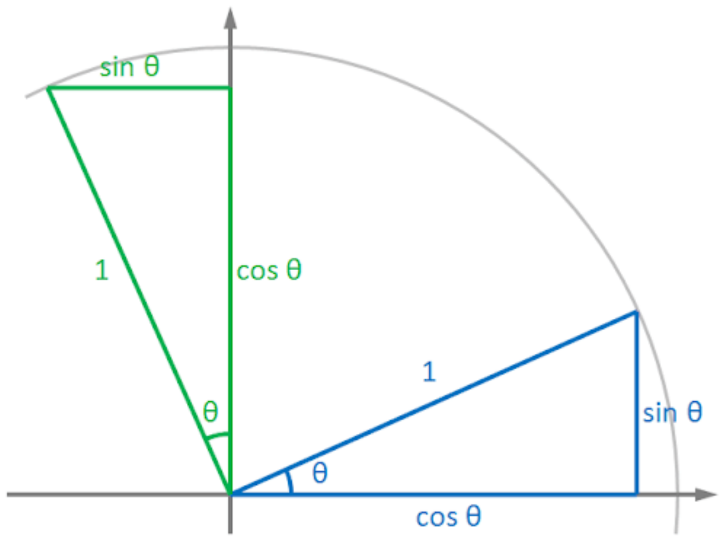
2 - D Rotation

In[ ]:= `Clear["Global`*"]`

In[ ]:=



In[ ]:=
```
m = {{Cos[θ], -Sin[θ]}, {+Sin[θ], Cos[θ]}};
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

$$a = \left\{\left\{-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$$

In[ ]:= `ev = Normalize /@ Eigenvectors[m]`

Out[ ]= $\left\{\left\{-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$

In[ ]:= `ev.m.Transpose[ev] // Simplify // TrigToExp // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$$

In[ ]:= `ev.m.Transpose[ev] // Simplify // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & \text{Cos}[x] + i\,\text{Sin}[x] \\ \text{Cos}[x] - i\,\text{Sin}[x] & 0 \end{pmatrix}$$

In[ ]:=
```
Transpose[ev].m.ev // Simplify // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} -\text{Cos}[x] & i\,\text{Sin}[x] \\ -i\,\text{Sin}[x] & \text{Cos}[x] \end{pmatrix}$$

In[ ]:=
```
vec = {a, b};
vec // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

In[ ]:=
```
m.vec // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} a\,\text{Cos}[x] - b\,\text{Sin}[x] \\ b\,\text{Cos}[x] + a\,\text{Sin}[x] \end{pmatrix}$$

In[ ]:=
```
xVec = {1, 0};
yVec = {0, 1};
m.xVec // MatrixForm
m.yVec // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[x] \\ \text{Sin}[x] \end{pmatrix}$$

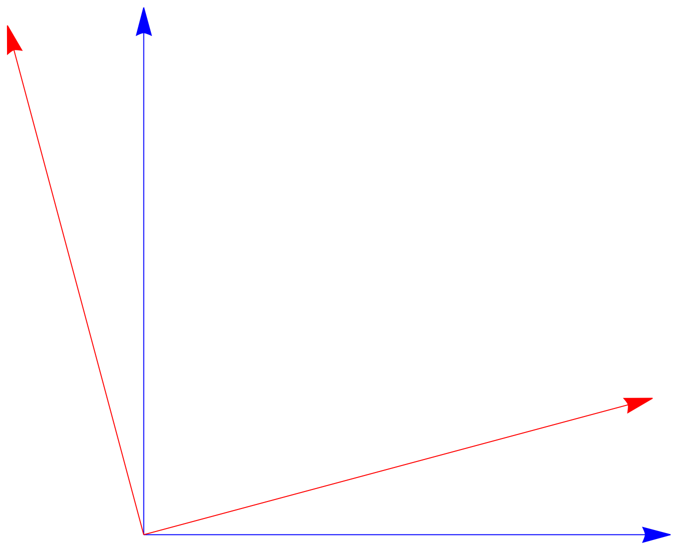
Out[ ]//MatrixForm=

$$\begin{pmatrix} -\text{Sin}[x] \\ \text{Cos}[x] \end{pmatrix}$$

In[ ]:=
```
makeVec[v_] := Arrow[{{0, 0}, v}]
```
In[ ]:=
```
doPlot[θ_] :=
 {Blue, makeVec[xVec], makeVec[yVec], Red, makeVec[m.xVec], makeVec[m.yVec]} /.
   {x → θ Degree} // Graphics
```

In[ ]:= doPlot[15]

Out[ ]=



2 - D Eigenvalues and Eigenvectors

In[ ]:=
```
evals = Eigenvalues[m];
evals // TrigToExp
```

Out[ ]= $\{e^{-i x}, e^{i x}\}$

In[ ]:=
```
evecs = Eigenvectors[m]
```

Out[ ]= {{-i, 1}, {i, 1}}

In[ ]:=
```
m.evecs[[1]]
```

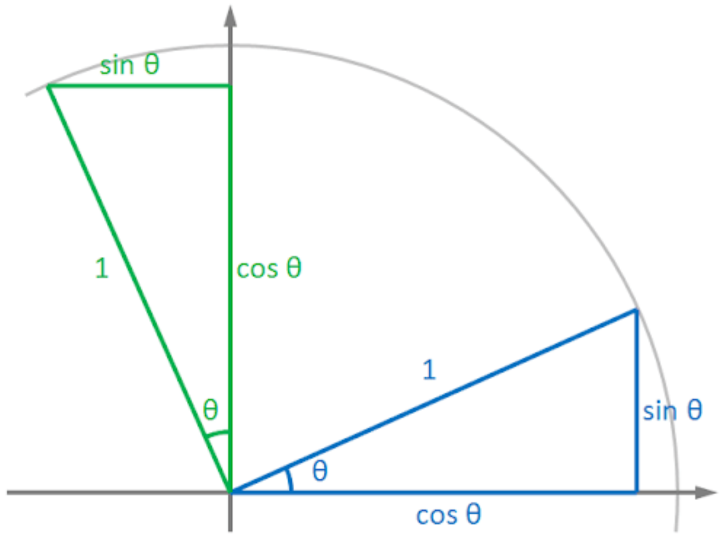
Out[ ]= {-i Cos[x] - Sin[x], Cos[x] - i Sin[x]}

In[ ]:=
```
evals[[1]] × evecs[[1]] // Simplify
```

Out[ ]= {-i Cos[x] - Sin[x], Cos[x] - i Sin[x]}

3 - D Rotation

In[ ]:=
```
Clear["Global`*"]
```



In[◦]:=
```
m = {{Cos[x], -Sin[x], 0}, {+Sin[x], Cos[x], 0}, {0, 0, 1}};
m // MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[x] & -\text{Sin}[x] & 0 \\ \text{Sin}[x] & \text{Cos}[x] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[◦]:=
```
vec = {a, b, c};
vec // MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In[◦]:=
```
m.vec // MatrixForm
```

Out[◦]//MatrixForm=

$$\begin{pmatrix} a\,\text{Cos}[x] - b\,\text{Sin}[x] \\ b\,\text{Cos}[x] + a\,\text{Sin}[x] \\ c \end{pmatrix}$$

In[ ]:=
```
xVec = {1, 0, 0};
yVec = {0, 1, 0};
zVec = {0, 0, 1}
m.xVec // MatrixForm
m.yVec // MatrixForm
m.zVec // MatrixForm
```

Out[ ]= {0, 0, 1}

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[x] \\ \text{Sin}[x] \\ 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\text{Sin}[x] \\ \text{Cos}[x] \\ 0 \end{pmatrix}$$

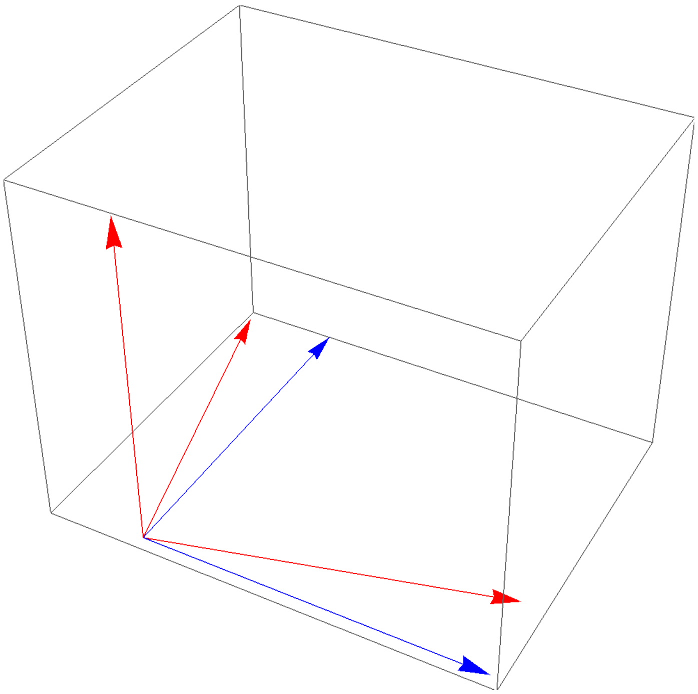
Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In[ ]:=
```
makeVec[v_] := Arrow[{{0, 0, 0}, v}]
```

In[ ]:=
```
Clear[doPlot]
doPlot[θ_] := {Blue, makeVec[xVec], makeVec[yVec], makeVec[zVec], Red,
    makeVec[m.xVec], makeVec[m.yVec], makeVec[m.zVec]} /. {x → θ Degree} // Graphics3D
```

In[ ]:= doPlot[15]

Out[ ]=


3 - D Eigenvalues and Eigenvectors

In[ ]:=
```
evals = Eigenvalues[m];
evals // TrigToExp
```

Out[ ]= $\{1, e^{-i x}, e^{i x}\}$

In[ ]:=
```
evecs = Eigenvectors[m]
```

Out[ ]= {{0, 0, 1}, {-i, 1, 0}, {i, 1, 0}}

In[ ]:=
```
m.evecs[[1]]
```

Out[ ]= {0, 0, 1}

In[ ]:=
```
evals[[1]] × evecs[[1]] // Simplify
```

Out[ ]= {0, 0, 1}

Stretch

In[ ]:=
```
Clear["Global`*"]
```

In[ ]:=
```
m = DiagonalMatrix [{1, 2}];
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

In[ ]:=
```
Det[m]
```

Out[ ]= 2

In[ ]:=
```
vec = {a, b};
vec // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

In[ ]:=
```
m.vec // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} a \\ 2\,b \end{pmatrix}$$

In[ ]:=
```
xVec = {1, 0};
yVec = {0, 1};
m.xVec // MatrixForm
m.yVec // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

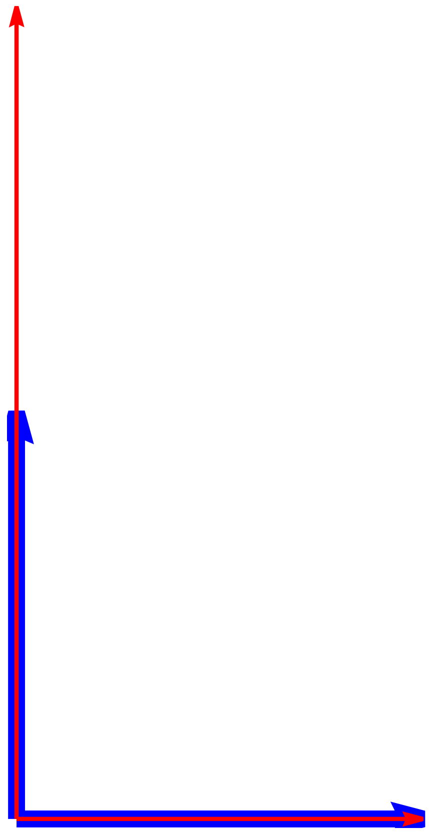
Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

In[ ]:=
```
makeVec[v_] := Arrow[{{0, 0}, v}]
```

In[ ]:=
```
Clear[doPlot]
doPlot[θ_] := {Thickness[0.04], Blue, makeVec[xVec], makeVec[yVec], Thickness[0.01],
    Red, makeVec[m.xVec], makeVec[m.yVec]} /. {x → θ Degree} // Graphics
doPlot[
 15]
```

Out[ ]=



Rotated Stretch

Stretch & Rotate

In[ ]:=
```
Clear["Global`*"]
```

In[ ]:=
```
m = {{5/4, -√3/4}, {-√3/4, 7/4}};
m // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{5}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{7}{4} \end{pmatrix}$$

In[ ]:=
```
Det[m]
```

Out[ ]= 2

In[ ]:=
```
Eigenvalues[m]
```

Out[ ]= $\{2, 1\}$

In[ ]:=
```
ev = Eigenvectors[m]
```

Out[ ]= $\left\{\left\{-\frac{1}{\sqrt{3}}, 1\right\}, \left\{\sqrt{3}, 1\right\}\right\}$

In[ ]:=
```
makeVec[v_] := Arrow[{{0, 0}, v}]
```

In[ ]:=
```
ev1 = ev[[1]] // Normalize
ev2 = ev[[2]] // Normalize

ev1p = m.ev[[1]] // Simplify
ev2p = m.ev[[2]]
```

Out[ ]= $\left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}\right\}$

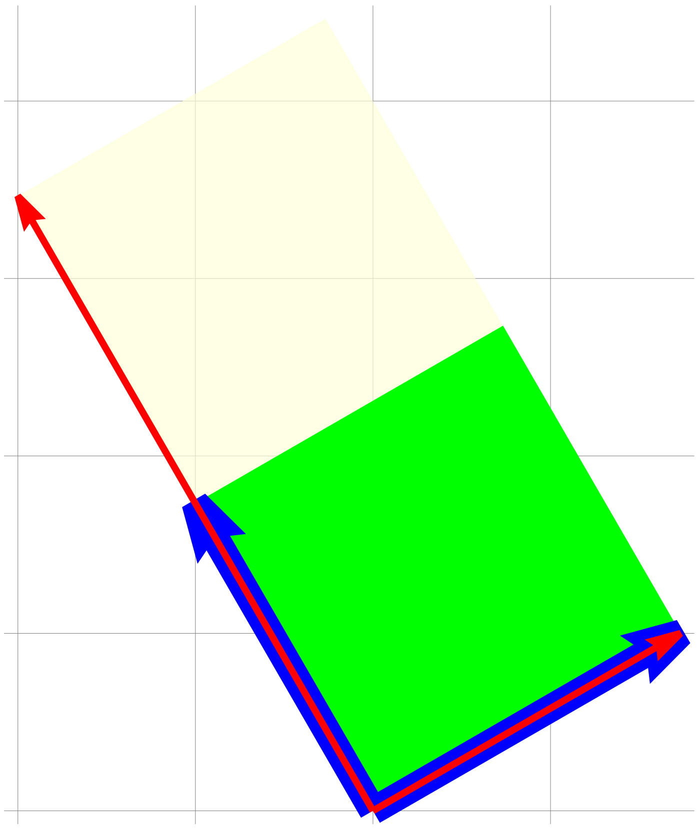
Out[ ]= $\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}$

Out[ ]= $\left\{-\frac{2}{\sqrt{3}}, 2\right\}$

Out[ ]= $\left\{\sqrt{3}, 1\right\}$

In[ ]:=
```
Graphics[{
  Opacity[0.7], LightYellow, Rotate[Rectangle[{0, 0}, {1, 2}], π / 6, {0, 0}],
  Opacity[1.0], Green, Rotate[Rectangle[{0, 0}, {1, 1}], π / 6, {0, 0}],
  GridLines → Automatic, Thickness[0.04],
  Blue, makeVec[ev1], makeVec[ev2], Thickness[0.01],
  Red, makeVec[m.ev1], makeVec[m.ev2]},
 GridLines → Automatic]
```

Out[ ]=



Un-Rotate

In[ ]:=
```
ev = Normalize /@ Eigenvectors[m];
ev // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

In[ ]:=
```
ev.m.Transpose[ev] // MatrixForm
```

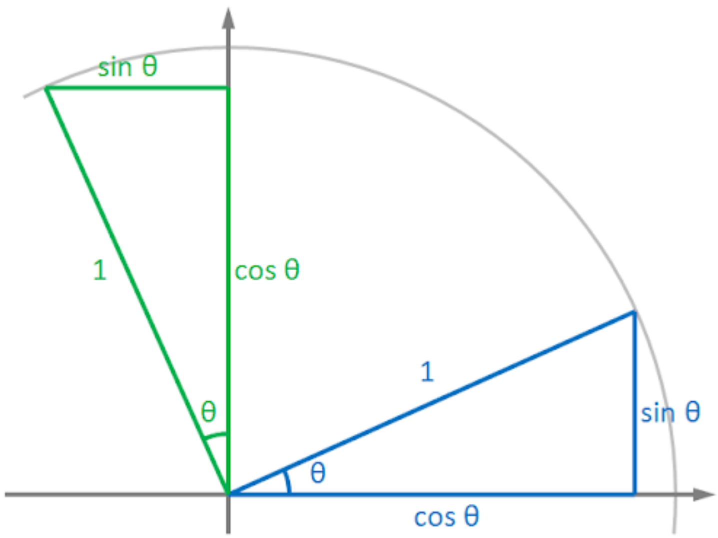
Out[ ]//MatrixForm=

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

2 - D Rotation Angle:

In[ ]:=
```
Clear["Global`*"]
```

In[ ]:=



In[ ]:=
```
m[x_] = {{Cos[x], -Sin[x]}, {+Sin[x], Cos[x]}};
m[x] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[x] & -\text{Sin}[x] \\ \text{Sin}[x] & \text{Cos}[x] \end{pmatrix}$$

In[ ]:=
```
Tr[m[x]]
```

Out[ ]= 2 Cos[x]

In[ ]:=
```
m[π / 3] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

In[ ]:=
```
eq = 2 Cos[θ] == Tr[ m[π / 3] ]
```

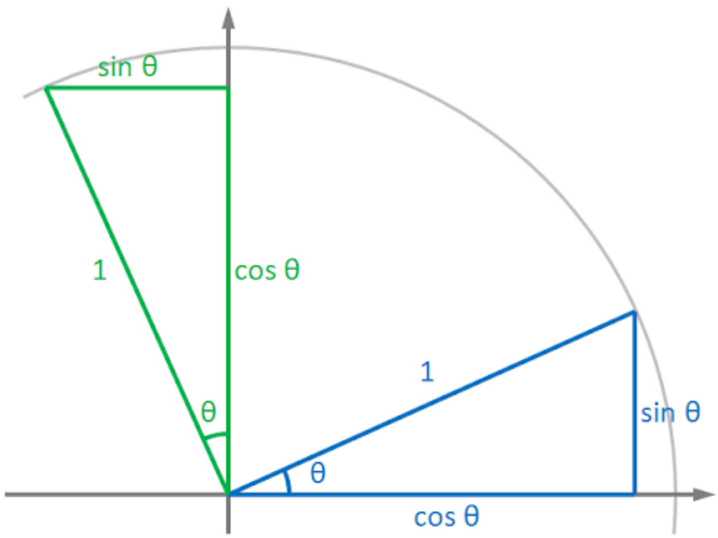
Out[ ]= 2 Cos[θ] == 1

In[ ]:=
```
Solve[eq, θ] /. {C[_] → 0} // Normal
```

Out[ ]= $\left\{\left\{\theta \to -\frac{\pi}{3}\right\}, \left\{\theta \to \frac{\pi}{3}\right\}\right\}$

3 - D Rotation Angle

In[ ]:=
```
Clear["Global`*"]
```



In[ ]:=
```
m[x_] = {{Cos[x], -Sin[x], 0}, {+Sin[x], Cos[x], 0}, {0, 0, 1}};
m[x] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[x] & -\text{Sin}[x] & 0 \\ \text{Sin}[x] & \text{Cos}[x] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
Tr[m[x]]
```

Out[ ]= 1 + 2 Cos[x]

In[ ]:=
```
m[π / 3] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
eq = Tr[ m[π / 3]] == 2 Cos[θ] + 1
```

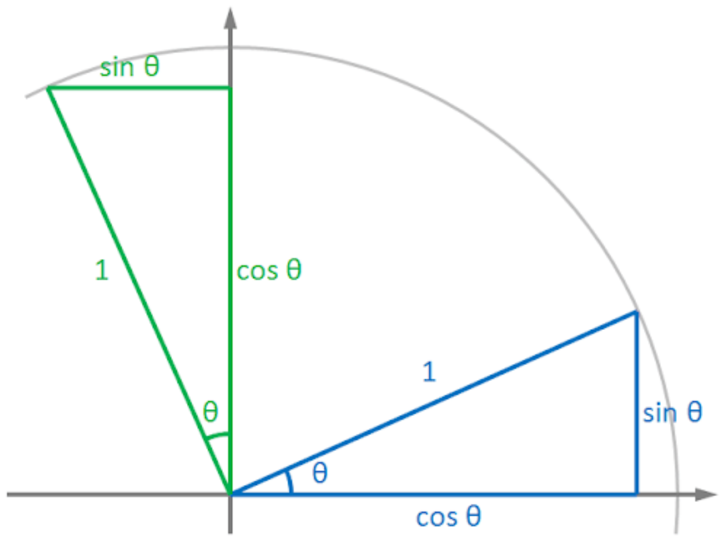
Out[ ]= 2 == 1 + 2 Cos[θ]

In[ ]:=
```
Solve[eq, θ] /. {C[_] → 0} // Normal
```

Out[ ]= $\left\{\left\{\theta \to -\frac{\pi}{3}\right\}, \left\{\theta \to \frac{\pi}{3}\right\}\right\}$

Euler Angles

In[ ]:=
```
Clear["Global`*"]
```



In[ ]:=
```
mx[θ_] = {{1, 0, 0}, {0, Cos[θ], -Sin[θ]}, {0, +Sin[θ], Cos[θ]}};
my[θ_] = {{Cos[θ], 0, +Sin[θ]}, {0, 1, 0}, {-Sin[θ], 0, Cos[θ]}};
mz[θ_] = {{Cos[θ], -Sin[θ], 0}, {+Sin[θ], Cos[θ], 0}, {0, 0, 1}};
mx[θ] // MatrixForm
my[θ] // MatrixForm
mz[θ] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\theta] & -\text{Sin}[\theta] \\ 0 & \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] & 0 \\ \text{Sin}[\theta] & \text{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
euler = mz[θ1].mx[θ2].mz[θ3];
euler // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta1]\,\text{Cos}[\theta3] - \text{Cos}[\theta2]\,\text{Sin}[\theta1]\,\text{Sin}[\theta3] & -\text{Cos}[\theta2]\,\text{Cos}[\theta3]\,\text{Sin}[\theta1] - \text{Cos}[\theta1]\,\text{Sin}[\theta3] & \text{Sin}[\theta1]\,\text{Sin}[\theta2] \\ \text{Cos}[\theta3]\,\text{Sin}[\theta1] + \text{Cos}[\theta1]\,\text{Cos}[\theta2]\,\text{Sin}[\theta3] & \text{Cos}[\theta1]\,\text{Cos}[\theta2]\,\text{Cos}[\theta3] - \text{Sin}[\theta1]\,\text{Sin}[\theta3] & -\text{Cos}[\theta1]\,\text{Sin}[\theta2] \\ \text{Sin}[\theta2]\,\text{Sin}[\theta3] & \text{Cos}[\theta3]\,\text{Sin}[\theta2] & \text{Cos}[\theta2] \end{pmatrix}$$

In[ ]:=
```
Det[euler] // Simplify
```

Out[ ]= 1

```
In[ ]:= Eigenvalues[euler] // Simplify
```

Out[ ]= $\left\{1,\ \frac{1}{4}\left(-2+2\,\text{Cos}[\theta 2]+2\,\text{Cos}[\theta 1+\theta 3]+\text{Cos}[\theta 1-\theta 2+\theta 3]+\text{Cos}[\theta 1+\theta 2+\theta 3]-\sqrt{\left(-16+\left(-2+2\,\text{Cos}[\theta 2]+2\,\text{Cos}[\theta 1+\theta 3]+\text{Cos}[\theta 1-\theta 2+\theta 3]+\text{Cos}[\theta 1+\theta 2+\theta 3]\right)^2\right)}\right),\ \frac{1}{4}\left(-2+2\,\text{Cos}[\theta 2]+2\,\text{Cos}[\theta 1+\theta 3]+\text{Cos}[\theta 1-\theta 2+\theta 3]+\text{Cos}[\theta 1+\theta 2+\theta 3]+\sqrt{\left(-16+\left(-2+2\,\text{Cos}[\theta 2]+2\,\text{Cos}[\theta 1+\theta 3]+\text{Cos}[\theta 1-\theta 2+\theta 3]+\text{Cos}[\theta 1+\theta 2+\theta 3]\right)^2\right)}\right)\right\}$

```
In[ ]:= tmp = Eigensystem[euler];
        Dimensions[tmp]
Out[ ]= {2, 3}

In[ ]:= tmp[[1, 1]]
Out[ ]= 1

In[ ]:= eVec = tmp[[2, 1]]
```

Out[ ]= $\{-\text{Cot}[\theta 2]\,\text{Csc}[\theta 3]+\text{Csc}[\theta 2]\,\text{Csc}[\theta 3]+\left(\text{Cot}[\theta 3]\,\text{Csc}[\theta 2]\left(-\text{Cos}[\theta 3]\,\text{Sin}[\theta 1]+\text{Cos}[\theta 2]\,\text{Cos}[\theta 3]\,\text{Sin}[\theta 1]-\text{Cos}[\theta 1]\,\text{Cos}[\theta 2]\,\text{Sin}[\theta 3]+\text{Cos}[\theta 1]\,\text{Cos}[\theta 2]^2\,\text{Sin}[\theta 3]+\text{Cos}[\theta 1]\,\text{Sin}[\theta 2]^2\,\text{Sin}[\theta 3]\right)\right)/\left(\text{Cos}[\theta 3]^2\,\text{Sin}[\theta 1]+\text{Sin}[\theta 3]+\text{Sin}[\theta 1]\,\text{Sin}[\theta 3]^2\right),\ -\left(\left(\text{Csc}[\theta 2]\left(-\text{Cos}[\theta 3]\,\text{Sin}[\theta 1]+\text{Cos}[\theta 2]\,\text{Cos}[\theta 3]\,\text{Sin}[\theta 1]-\text{Cos}[\theta 1]\,\text{Cos}[\theta 2]\,\text{Sin}[\theta 3]+\text{Cos}[\theta 1]\,\text{Cos}[\theta 2]^2\,\text{Sin}[\theta 3]+\text{Cos}[\theta 1]\,\text{Sin}[\theta 2]^2\,\text{Sin}[\theta 3]\right)\right)/\left(\text{Cos}[\theta 3]^2\,\text{Sin}[\theta 1]+\text{Sin}[\theta 3]+\text{Sin}[\theta 1]\,\text{Sin}[\theta 3]^2\right)\right),\ 1\}$

```
In[ ]:= euler.eVec - eVec // Simplify
Out[ ]= {0, 0, 0}
```