Calculus of Variations

Action $= S = \int_a^b L(q\,\mathring{q})\,dt$

Find path that minimizes S

$$\delta S = 0 = \delta \int_a^b L(q\,\mathring{q})\,dt$$

$$0 = \int_a^b \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \mathring{q}}\delta \mathring{q}$$

$$0 = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \mathring{q}}\right)\delta q$$

Integration by parts:

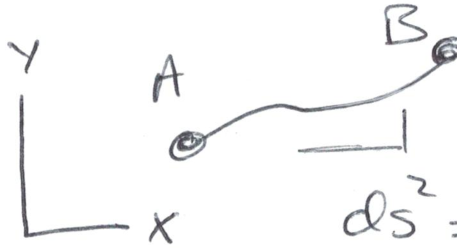
$$\int d(uv) = \int (du)\,v + u\,(dv)$$

$uv\Big]_a^b$ Assume $= 0$

Minus Sign

Thus $(du)\,v = -u\,(dv)$

Example 6.1



$$ds^2 = dx^2 + dy^2$$

$$\text{Length} = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{Length} = \int_A^B \sqrt{1 + \dot{y}^2}\, dx = \int_A^B F(y, \dot{y})\, dx$$

This is in the form for Euler-Lagrange

$$\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = 0$$

$$\frac{\partial F}{\partial y} = 0 \quad \Rightarrow \quad -\frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = 0 \Rightarrow \frac{\partial F}{\partial \dot{y}} = \text{const}$$

$$\frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{const} \quad \Rightarrow \quad \dot{y} = \text{const}$$

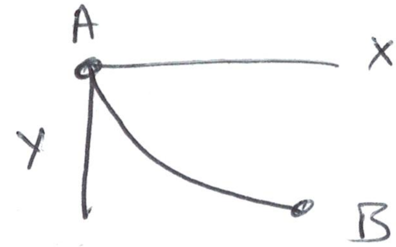
$$\Rightarrow \quad y = mx + b$$

$$\dot{y} = m = \frac{dy}{dx}$$

$$\int dy = \int dx \, m$$

$$y = mx + b$$

Brachistochrone Problem:



$$x = v t \qquad t = \frac{x}{v}$$

$$v = \sqrt{2gy} \quad \text{Since } \frac{1}{2} m v^2 = mgy$$

$$\text{time} = \int_A^B dt = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \int \frac{\sqrt{\frac{dx^2}{dy^2} + 1}}{\sqrt{2gy}}\, dy$$

$$\text{time} = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{\dot{x}^2 + 1}}{\sqrt{y}}\, dy = \frac{1}{\sqrt{2g}} \int F(x, \dot{x})\, dy$$

Now this is in the form of the Euler-Lagrange

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial \dot{x}} = 0 \qquad \text{but } \frac{\partial F}{\partial x} = 0$$

$$\therefore \quad \frac{\partial F}{\partial \dot{x}} = \text{const} = \frac{1}{\sqrt{y}} \frac{\dot{x}}{\sqrt{\dot{x}^2 + 1}}$$

Square this for convenience.

$$\frac{\dot{x}^2}{y(1 + \dot{x}^2)} = \text{const} = \frac{1}{2a} \qquad \text{(Eq 6.22)}$$

SHO w/ Lagrange Eqs.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \qquad L = T - V$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{q}^2 \qquad V = \frac{1}{2} k x^2 = \frac{1}{2} k q^2$$

$$L = T - V \qquad \frac{\partial L}{\partial q} = -kq \qquad \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

Eqs. $\Rightarrow$

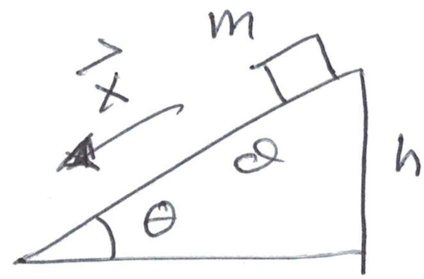
$$-kq - m\ddot{q} = 0$$

$$+m\ddot{q} + kq = 0$$

$$\ddot{q} + \omega_0^2 q = 0 \qquad \omega_0^2 = \frac{k}{m}$$

Mass on an incline:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$$



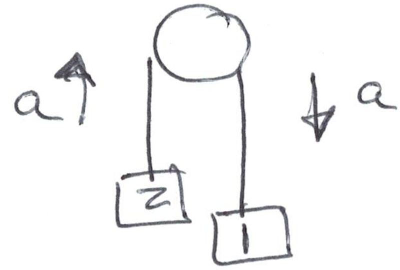
$$V = mgh = mg\, x \sin\theta$$

$$L = T - V \qquad \frac{\partial L}{\partial x} = -mg\sin\theta \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$Eq \Rightarrow -mg\sin\theta - m\ddot{x} = 0$$

$$\ddot{x} = -g\sin\theta$$

Atwood Machine :



$$m_1 g - T = m_1 a$$

$$m_2 g - T = -m_2 a$$

$$a = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right) \qquad T = g\left(\frac{2 m_1 m_2}{m_1 + m_2}\right)$$

Tension

Lagrange Eqs. $\quad x_1 + x_2 = L \Rightarrow x_2 = L - x_1$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \Rightarrow \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2$$

Kinetic Energy

$$-\dot{x}_1 = \dot{x}_2$$

$$V = m_1 g x_1 + m_2 g x_2 = m_1 g x_1 + m_2 g (L - x_1)$$

$$\mathcal{L} = T - V \qquad \frac{\partial L}{\partial x_1} = -m_1 g + m_2 g \qquad \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Rightarrow -(m_1 - m_2) g - (m_1 + m_2)\ddot{x}_1 = 0$$

$$-\ddot{x} = g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

Lagrange Multiplier

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = \lambda \frac{\partial F}{\partial q}}$$

Constraint Equation $\quad F = x_1 + x_2 - L = 0$

$$\frac{\partial F}{\partial x_1} = +1 \qquad \frac{\partial F}{\partial x_2} = +1 \qquad \text{Note } \ddot x_1 = -\ddot x_2$$

$$T = \frac{m_1}{2}\dot x_1^2 + \frac{m_2}{2}\dot x_2^2 \qquad V = m_1 g x_1 + m_2 g x_2 \qquad L = T - V$$

$$\frac{\partial L}{\partial x_1} = -m_1 g \qquad \frac{\partial L}{\partial x_2} = -m_2 g \qquad \frac{\partial L}{\partial \dot x_1} = m_1 \dot x_1 \qquad \frac{\partial L}{\partial \dot x_2} = m_2 \dot x_2$$

$$\left.\begin{aligned} &(1) \Rightarrow -m_1 g - m_1 \ddot x_1 = \lambda \\ &(2) \Rightarrow -m_2 g - m_2 \ddot x_2 = \lambda \end{aligned}\right\} \text{ with } -\ddot x_1 = \ddot x_2$$

$$(1) - (2) \Rightarrow -(m_1 - m_2) g - (m_1 + m_2)\ddot x = 0$$

$$\ddot x_1 = -\ddot x_2 = -g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$\lambda = -g\left(\frac{2 m_1 m_2}{m_1 + m_2}\right) \simeq \text{Tension}$$