Phys 3344: Thursday 29 Oct

Office Hours: Wed 5:00-6:00

Grades:

- scaled
- make up homework promptly

Homework #10:

- MMA Starter

Ch 11

Ch 13

	2020 FALL PHYS 3344				
#	DAY	LECTURE:	NOTES:	Chpt	TOPIC
1	TUE	08/25/20	First Class	1	Newtons Laws
2	THUR	08/27/20		2	Projectiles
3	TUE	09/01/20		3	Momentum & Angular Momentum
4	THUR	09/03/20		4	Energy
5	TUE	09/08/20		5	Oscillations
6	THUR	09/10/20			
7	TUE	09/15/20			
8	THUR	09/17/20			**EXAM 1**
9	TUE	09/22/20		6	Calculus of Variations
10	THUR	09/24/20		7	Lagrange's Equation
11	TUE	09/29/20			
12	THUR	10/01/20		8	Two Body Problems
13	TUE	10/06/20			
14	THUR	10/08/20		9	Non-Inertial Frames
	TUE	10/13/20	~~Fall Break~~	10	Rotational Motion
15	THUR	10/15/20			**EXAM 2**
16	TUE	10/20/20		10	Rotational Motion
17	THUR	10/22/20			
18	TUE	10/27/20		11	Coupled Oscillations
19	THUR	10/29/20			
20	TUE	11/03/20		13	Hamiltonian Mechanics
21	THUR	11/05/20	Drop Date		
22	TUE	11/10/20			
23	THUR	11/12/20			**EXAM 3**
24	TUE	11/17/20		14	Collision Theory
25	THUR	11/19/20			
26	TUE	11/24/20		15	Special relativity
27	THUR	11/26/20	**Thanksgiving**		No Class
28	TUE	12/01/20			No Class
29	THUR	12/03/20	**Last Class**		Review
	WED	**Dec 16**	**FINAL EXAM**	**Wednesday Dec. 16,2020, 11:30am – 2:30**	
	Adjustments may be made depending on student interests/needs and unplanned events				

Chapter 11 Coupled Motion

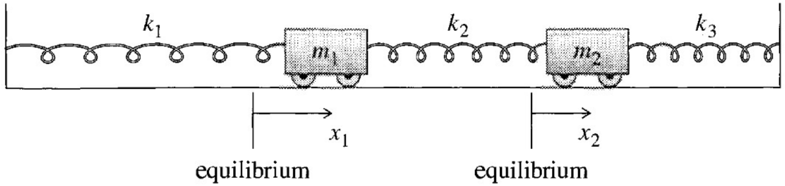

Figure 11.1 Two carts attached to fixed walls by the springs labeled k_1 and k_3, and to each other by k_2. The carts' positions x_1 and x_2 are measured from their respective equilibrium positions.

11.2 Identical Springs and Equal Masses

Let us continue to examine the two carts of Figure 11.1, but suppose now that the two masses are equal, $m_1 = m_2 = m$, and similarly the three spring constants, $k_1 = k_2 = k_3 = k$. In this case, the matrices **M** and **K** defined in (11.5) reduce to

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}. \tag{11.13}$$

The matrix $(\mathbf{K} - \omega^2\mathbf{M})$ of the generalized[3] eigenvalue equation (11.11) becomes

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \tag{11.14}$$

and its determinant is

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = (2k - m\omega^2)^2 - k^2 = (k - m\omega^2)(3k - m\omega^2).$$

The two normal freqencies are determined by the condition that this determinant be zero and are therefore

$$\omega = \sqrt{\frac{k}{m}} = \omega_1 \quad \text{and} \quad \omega = \sqrt{\frac{3k}{m}} = \omega_2. \tag{11.15}$$

These two normal frequencies are the frequencies at which our two carts can oscillate in purely sinusoidal motion. Notice that the first one, ω_1, is precisely the frequency of a single mass m on a single spring k. We shall see the reason for this apparent coincidence in a moment.

[2] Since there are two solutions for ω^2, you might think this would give four solutions for $\omega = \pm\sqrt{\omega^2}$. However, a glance at Equations (11.6) and (11.7) will convince you that $+\omega$ and $-\omega$ constitute the *same* frequency for the real motion.

[3] From now on, I shall refer to (11.11) as the eigenvalue equation, omitting the "generalized."

Equation (11.15) tells us the two possible frequencies of our system, but we have not yet described the corresponding motions. Recall that the actual motion is given by the column of real numbers $\mathbf{x}(t) = \text{Re}\,\mathbf{z}(t)$ where the complex column $\mathbf{z}(t) = \mathbf{a}e^{i\omega t}$, and **a** is made up of two fixed numbers,

$$\mathbf{a} = \begin{bmatrix} a_1, \\ a_2 \end{bmatrix},$$

which must satisfy the eigenvalue equation

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0. \tag{11.16}$$

Now that we know the possible normal frequencies, we must solve this equation for the vector **a** for each normal frequency in turn. The sinusoidal motion with any one of the normal frequencies is called a **normal mode**, and I shall start with the first normal mode.

The First Normal Mode

If we choose ω equal to the first normal frequency, $\omega_1 = \sqrt{k/m}$, then the matrix $(\mathbf{K} - \omega^2\mathbf{M})$ of (11.14) becomes

$$(\mathbf{K} - \omega_1^2\mathbf{M}) = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}. \tag{11.17}$$

(Notice that this matrix has determinant 0, as it should.) Therefore, for this case, the eigenvalue equation (11.16) reads

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

which is equivalent to the two equations

$$a_1 - a_2 = 0$$
$$-a_1 + a_2 = 0.$$

Notice that these two equations are actually the same equation, and either one implies that $a_1 = a_2 = Ae^{-i\delta}$, say. The complex column $\mathbf{z}(t)$ is therefore

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)}$$

and the corresponding actual motion is given by the real column $\mathbf{x}(t) = \text{Re}\,\mathbf{z}(t)$ or

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta).$$

That is,

$$\left.\begin{aligned} x_1(t) &= A\cos(\omega_1 t - \delta) \\ x_2(t) &= A\cos(\omega_1 t - \delta) \end{aligned}\right\} \quad \text{[first normal mode]}. \tag{11.18}$$

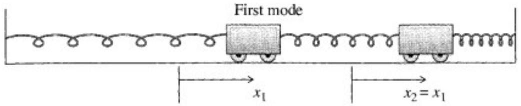

Figure 11.2 The first normal mode for two equal-mass carts with three identical springs. The two carts oscillate back and forth with equal amplitudes and exactly in phase, so that $x_1(t) = x_2(t)$, and the middle spring remains at its equilibrium length all the time.

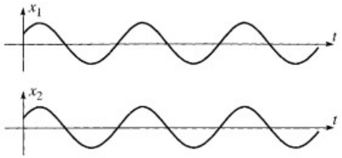

Figure 11.3 In the first mode, the two positions oscillate sinusoidally, with equal ampitudes and in phase.

We see that in the first normal mode the two carts oscillate in phase and with the same amplitude A, as shown in Figure 11.2.

A striking feature of Figure 11.2 is that, because $x_1(t) = x_2(t)$, the middle spring is neither stretched nor compressed during the oscillations. This means that, for the first normal mode, the middle spring is actually irrelevant, and each cart oscillates just as if it were attached to a single spring. This explains why the first normal frequency $\omega_1 = \sqrt{k/m}$ is the same as for a single cart on a single spring.

Another way to illustrate the motion in the first normal mode is just to plot the two positions x_1 and x_2 as functions of t. This is shown in Figure 11.3.

The Second Normal Mode

The second normal frequency at which our system can oscillate sinusoidally is given by (11.15) as $\omega_2 = \sqrt{3k/m}$, which, when substituted into (11.14), gives

$$(\mathbf{K} - \omega_2^2\mathbf{M}) = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}. \tag{11.19}$$

Thus, for this normal mode, the eigenvalue equation $(\mathbf{K} - \omega_2^2\mathbf{M})\mathbf{a} = 0$ implies that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

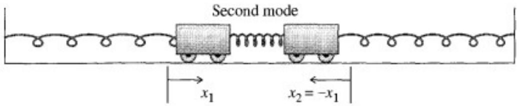

Figure 11.4 The second normal mode for two equal-mass carts with three identical springs. The two carts oscillate back and forth with equal amplitudes but exactly out of phase, so that $x_2(t) = -x_1(t)$ at all times.

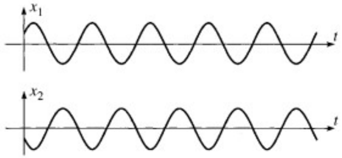

Figure 11.5 In the second mode, the two positions oscillate sinusoidally, with equal ampitudes but exactly out of phase.

which implies that $a_1 + a_2 = 0$, or $a_1 = -a_2 = Ae^{-i\delta}$, say. The complex column $\mathbf{z}(t)$ is therefore

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_2 t} = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)}$$

and the corresponding actual motion is given by the real column $\mathbf{x}(t) = \text{Re}\,\mathbf{z}(t)$ or

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_2 t - \delta).$$

That is,

$$\left.\begin{aligned} x_1(t) &= A\cos(\omega_2 t - \delta) \\ x_2(t) &= -A\cos(\omega_2 t - \delta) \end{aligned}\right\} \quad \text{[second normal mode]}. \tag{11.20}$$

We see that in the second normal mode the two carts oscillate with the same amplitude A but exactly out of phase, as shown in the picture of Figure 11.4 and the graphs of Figure 11.5.

Notice that in the second normal mode, when cart 1 is displaced to the right, cart 2 is displaced an equal distance to the left, and vice versa. This means that when the outer two springs are stretched (as in Figure 11.4), the middle spring is compressed by twice as much. Thus, for example, when the left spring is pulling cart 1 to the left, the middle spring is pushing cart 1, also to the left, with a force that is twice as large. This means that each cart moves as if it were attached to a single spring with force constant $3k$. In particular, the second normal frequency is $\omega_2 = \sqrt{3k/m}$.

The General Solution

We have now found two normal-mode solutions, which we can rewrite as

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \mathbf{x}(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

where ω_1 and ω_2 are the normal frequencies (11.15). Both of these solutions satisfy the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$ for any values of the four real constants A_1, δ_1, A_2, and δ_2. Because the equation of motion is linear and homogeneous, the sum of these two solutions is also a solution:

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2). \tag{11.21}$$

Because the equation of motion is really two second-order differential equations for the two variables $x_1(t)$ and $x_2(t)$, its general solution has four constants of integration. Therefore the solution (11.21), with its four arbitrary constants, is in fact the general solution. *Any* solution can be written in the form (11.21), with the constants A_1, A_2, δ_1, and δ_2 determined by the initial conditions.

The general solution (11.21) is hard to visualize and describe. The motion of each cart is a mixture of the two frequencies, ω_1 and ω_2. Since $\omega_2 = \sqrt{3}\omega_1$ the motion never repeats itself, except in the special case that one of the constants A_1 or A_2 is zero (which gives us back one of the normal modes). Figure 11.6 shows graphs of the two positions in a typical nonnormal mode (with $A_1 = 1$, $A_2 = 0.7$, $\delta_1 = 0$, and $\delta_2 = \pi/2$). About the only simple thing one can say about these graphs is that they certainly are not very simple!

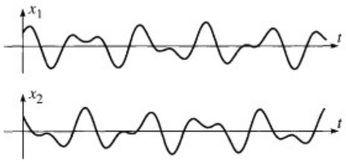

Figure 11.6 In the general solution, both $x_1(t)$ and $x_2(t)$ oscillate with *both* of the normal frequencies, producing a quite complicated nonperiodic motion.

Normal Coordinates

We have seen that in any possible motion of our two-cart system, both of the coordinates $x_1(t)$ and $x_2(t)$ vary with time. In the normal modes, their time dependence is simple (sinusoidal), but it is still true that both vary, reflecting that the two carts are coupled and that one cart cannot move without the other. It is possible to introduce alternative, so-called **normal coordinates** which, although less physically transparent, have the convenient property that each can vary independently of

the other. This statement is true for any system of coupled oscillators, but is especially easy to see in the present case of two equal masses joined by three identical springs.

In place of the coordinates x_1 and x_2, we can characterize the positions of the two carts by the two *normal coordinates*

$$\xi_1 = \tfrac{1}{2}(x_1 + x_2) \tag{11.22}$$

and

$$\xi_2 = \tfrac{1}{2}(x_1 - x_2). \tag{11.23}$$

The physical significance of the original variables x_1 and x_2 (as the positions of the two carts) is obviously more transparent, but ξ_1 and ξ_2 serve just as well to label the configuration of the system. Moreover, if you refer back to (11.18) for the first normal mode, you will see that in the first mode the new variables are given by

$$\left.\begin{array}{lcl} \xi_1(t) & = & A\cos(\omega_1 t - \delta) \\ \xi_2(t) & = & 0 \end{array}\right\} \quad \text{[first normal mode]}, \tag{11.24}$$

whereas in the second mode, we see from (11.20) that

$$\left.\begin{array}{lcl} \xi_1(t) & = & 0 \\ \xi_2(t) & = & A\cos(\omega_2 t - \delta) \end{array}\right\} \quad \text{[second normal mode]}. \tag{11.25}$$

In the first normal mode the new variable ξ_1 oscillates, but ξ_2 remains zero. In the second mode it is the other way round. In this sense, the new coordinates are independent — either can oscillate without the other. The general motion of our system is a superposition of both modes, and in this case both ξ_1 and ξ_2 oscillate, but ξ_1 oscillates at the frequency ω_1 only, and ξ_2 at the frequency ω_2 only. In some more complicated problems, these new normal coordinates represent a considerable simplification. (See Problems 11.9, 11.10, and 11.11 for some examples and Section 11.7 for further discussion.)

11.3 Two Weakly Coupled Oscillators

In the last section we discussed the oscillations of two equal masses joined by three equal springs. For this system, the two normal modes were easy to understand and to visualize, but the nonnormal oscillations were much less so. A system where some of the nonnormal oscillations are readily visualized is a pair of oscillators which have the same natural frequency and which are *weakly coupled*. As an example of such a system, consider the two identical carts shown in Figure 11.7, which are attached to their adjacent walls by identical springs (force constants k) and to each other by a much weaker spring (force constant $k_2 \ll k$).

We can quickly solve for the normal modes of this system. The mass matrix $\mathbf{M}$ is the same as before. The spring matrix $\mathbf{K}$ and the crucial combination $(\mathbf{K} - \omega^2\mathbf{M})$

11.7 ★★ [Computer] The most general motion of the two carts of Section 11.2 is given by (11.21), with the constants A_1, A_2, δ_1, and δ_2 determined by the initial conditions. **(a)** Show that (11.21) can be rewritten as

$$\mathbf{x}(t) = (B_1 \cos\omega_1 t + C_1 \sin\omega_1 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (B_2 \cos\omega_2 t + C_2 \sin\omega_2 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This form is usually a little more convenient for matching to given initial conditions. **(b)** If the carts are released from rest at positions $x_1(0) = x_2(0) = A$, find the coefficients B_1, B_2, C_1, and C_2 and plot $x_1(t)$ and $x_2(t)$. Take $A = \omega_1 = 1$ and $0 \le t \le 30$ for your plots. **(c)** Same as part (b), except that $x_1(0) = A$ but $x_2(0) = 0$.

Chapter 13 Hamiltonian Mechanics

Principal Definitions and Equations of Chapter 13

The Hamiltonian

If a system has generalized coordinates $\mathbf{q} = (q_1, \cdots, q_n)$, Lagrangian $\mathcal{L}$, and generalized momenta $p_i = \partial\mathcal{L}/\partial q_i$, its **Hamiltonian** is defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}, \qquad \text{[Eq. (13.22)]}$$

always considered as a function of the variables $\mathbf{q}$ and $\mathbf{p}$ (and possibly t).

Hamilton's Equations

The time evolution of a system is given by Hamilton's equations

$$\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \qquad [i = 1, \cdots, n]. \quad \text{[Eq. (13.25)]}$$