Special Relativity

Rotation leaves $\|v\|$ invariant

$$v' = R\,v \qquad \|v'\| = \|v\| \qquad R\,R^T = \mathbb{1}$$

$$v' = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx + sy \\ -sx + cy \end{pmatrix} \qquad \|v'\|^2 = x^2 + y^2 = \|v\|^2$$

metric $g = (1,1)$ diagonal

Boost $\quad v = (x, t) \qquad \|v\|^2 = t^2 - x^2 \equiv \|v'\|^2$

$$v' = B\,v = \begin{pmatrix} ch & sh \\ sh & ch \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} ct + sx \\ st + cx \end{pmatrix}$$

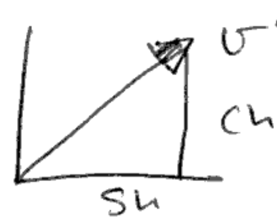
$$\|v'\|^2 = v' \circ g \circ v = (ct + sx)^2 - (st + cx)^2$$
$$= t^2 - x^2 \equiv \|v\|^2$$

Note: $\quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Boost $\quad B = \begin{pmatrix} ch & sh \\ sh & ch \end{pmatrix} \qquad ch^2 - sh^2 = 1$

If $\; v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad B\,v = \begin{pmatrix} c & s \\ s & h \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ s \end{pmatrix} = v'$

at rest $\qquad\qquad$ moving w/ velocity



$$\frac{v}{c} = \beta = \frac{\Delta x}{\Delta t} = \frac{sh}{ch} \Rightarrow sh = \beta\, ch$$

$$Sh = \beta\, ch \qquad ch^2 - sh^2 = 1 \quad \Rightarrow \quad ch = \gamma$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \therefore \quad ch = \gamma \quad sh = \gamma\beta$$

$$\text{Boost} = B = \begin{pmatrix} ch & sh \\ sh & ch \end{pmatrix} \equiv \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

$$v' = Bv = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(t+\beta x) \\ \gamma(x+\beta t) \end{pmatrix}$$

Check $\|v\|^2 = \|v'\|^2 = t^2 - x^2$

Other 4-vectors

$$p^\mu = (E, \vec{p}) \qquad A^\mu = (\phi, \vec{A}) \qquad j^\mu = (\rho, \vec{j})$$

4-Tensors

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & +B_z & 0 & -B_x \\ E_z & -B_y & +B_x & 0 \end{pmatrix}$$

Scattering

$$\xrightarrow{p_1} \xleftarrow{p_2}$$

CMS Frame

$$p_1^\mu = (E_1\ 0\ 0\ +p) \qquad p_1^2 = E_1^2 - p^2 \equiv m_1^2$$

$$p_2^\mu = (E_2\ 0\ 0\ -p) \qquad p_2^2 = E_2^2 - p^2 \equiv m_2^2$$

$$p_{12}^\mu = (E_1 + E_2\ \ 0\ 0\ 0) \qquad p_{12}^2 = (E_1+E_2)^2 \equiv s$$

invariants

Solve in terms of invariants $\{ m_1^2, m_2^2, s)$

$$E_{1,2} = \frac{s \pm m_1^2 \mp m_2^2}{2\sqrt{s}} \qquad p = \frac{\Delta(s, m_1^2, m_2^2)}{2\sqrt{s}}$$

$$\Delta^2(a,b,c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$$

Boost to Lab Frame $\quad B\, p_2 = (m_2\ 0\ 0\ 0)$

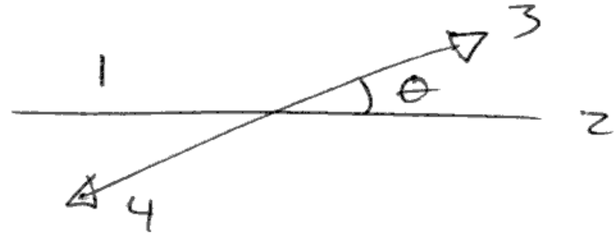
$$B\, p_2 = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} E_2 \\ p \end{pmatrix} = \begin{pmatrix} \gamma(E_2 - \beta p) \\ \gamma(-\beta E_2 + p) \end{pmatrix} = \begin{pmatrix} m_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \beta = \frac{p}{E_2} \quad \Rightarrow \gamma = \frac{E_2}{m_2} \quad \text{since } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$m_2^2 = E_2^2 - p^2$$

$$\gamma(E_2 - \beta p) = \frac{E_2}{m}\left(E_2 - \frac{p}{E_2} p\right)$$

$$= \frac{E_2^2 - p^2}{m} = m$$



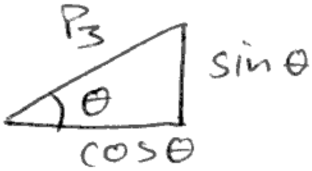
$$P_3^\mu = (E_3, \sin\theta\, p', 0, \cos\theta\, p') \qquad P_3^2 = m_3^2$$

$$P_4^\mu = (E_4, -\sin\theta\, p', 0, -\cos\theta\, p') \qquad P_4^2 = m_4^2$$

$$P_{34}^\mu = (E_3 + E_4, 0\ 0\ 0) \qquad P_{34}^2 = s$$

$$E_{3,4} = \frac{s \pm m_3^2 \mp m_4^2}{2\sqrt{s}} \qquad p' = \frac{\Delta(s, m_3^2, m_4^2)}{2\sqrt{s}}$$

Transform CM to Lab frame



$$\operatorname{Tan}\theta = \frac{P_{3x}}{P_{3z}} = \frac{p'\sin\theta}{p'\cos\theta} \equiv \tan\theta$$

Boost P_3 to lab frame: $\beta = \frac{P}{E_2}$ $\gamma = \frac{E_2}{m_2}$

$$\gamma\begin{pmatrix} 1 & +\beta \\ +\beta & 1 \end{pmatrix}\begin{pmatrix} E_3 \\ sP' \\ 0 \\ cP' \end{pmatrix} = \begin{pmatrix} \gamma(E_3 + \beta c P') \\ sP' \\ 0 \\ \gamma(+\beta E_3 + cP') \end{pmatrix} = \hat{P}_3$$

$$\operatorname{Tan}\theta' = \frac{\hat{P}_{3x}}{\hat{P}_{3z}} = \frac{sP'}{\gamma(+\beta E_3 + cP')}$$

IF Equal masses: $E_i \equiv E$, $m_i = m$, $P = p'$

$$\operatorname{Tan}\theta' = \frac{sP}{\gamma(+\beta E + cP)} = \frac{sP}{\gamma(+\frac{P}{E}E + cP)} = \frac{s}{\gamma(+1+c)}$$

$$\operatorname{Tan}\theta' = \frac{1}{\gamma}\operatorname{Tan}\left(\frac{\theta}{2}\right)$$

IF $\gamma \sim 1$ we get the classical result

$$\theta_{lab} \simeq \frac{\theta_{cm}}{2}$$

Eg $\theta_{cm} = 180^\circ$

$\theta_{lab} = 90^\circ$

CMS Energy for various Accelerators

Fermilab Fixed target   p → 1000 GeV   p $m = 1\ \text{GeV}$

$$P_1 \equiv (1000, 0, 0, 1000) = (E\ 0\ 0\ E)$$
$$P_2 = (1, 0\ 0\ 0) = (m\ 0\ 0\ 0)$$

$$P_{12} = (1001, 0, 0, 1000) = (E+m, 0, 0, E)$$

$$P_{12}^2 = (E+m)^2 - E^2 = 2Em + m^2 \cong 2Em$$
$$P_{12}^2 = 2000\ \text{GeV}^2 = s$$
$$\sqrt{s} = 45\ \text{GeV}$$

Collider mode

$$P_1 = (E\ 0\ 0\ E) \qquad E = 1000\ \text{GeV}$$
$$P_2 = (E\ 0\ 0\ -E)$$
$$P_{12} = (2E\ 0\ 0\ 0)$$

$$P_{12}^2 = s = 4E^2$$

$$\sqrt{s} = 2E = 2 \cdot 1000 = 2000\ \text{GeV}$$
$$= 2\ \text{TeV}$$

LHC: $E = 7000\ \text{GeV} = 7\ \text{TeV}$

$$\sqrt{s} = 14\ \text{TeV}$$

HERA: ep collider

$$\underset{820\ \text{GeV}}{\xrightarrow{p}} \qquad \underset{30\ \text{GeV}}{\xleftarrow{e}}$$

$$p = (p\ 0\ 0\ p) \qquad p \sim 820$$

$$e = (e\ 0\ 0\ {-e}) \qquad e \sim 30$$

$$P_{12} = (p+e,\ 0, 0,\ p-e)$$

$$P_{12}^2 = s = (p+e)^2 - (p-e)^2 = 4pe$$

$$s = 4(820)(30) = 98400\ \text{GeV}^2$$

$$\sqrt{s} = 2\sqrt{pe} = 314\ \text{GeV}$$

Produce $\bar{p}$ in collidermode

$$\xrightarrow{1} \xleftarrow{2}$$

$P_1 = (E\ 0\ 0\ P)$

$P_2 = (E\ 0\ 0\ -P)$

$P_{12} = (2E\ 0\ 0\ 0)$ $\quad P_{12}^2 = S = 4E^2 \quad \sqrt{s} = 2E$

After $pp \rightarrow ppp\bar{p}$

$P_3 = (m\ 0\ 0\ 0)$

$P_4 = (m\ 0\ 0\ 0)$

$P_5 = (m\ 0\ 0\ 0)$

$P_6 = (m\ 0\ 0\ 0)$

$P_{Final} = (4m, 0\ 0\ 0)$

$s = P_{Final}^2 = 16m^2 = 4E^2$

$4m^2 = E^2$

$2m = E$

Produce $\bar{p}$ in fixed target mode

$$P_1 = (E, 0\,0\,P)$$

$$P_2 = (m, 0\,0\,0)$$

$$P_{12} = (E+m, 0\,0\,P)$$

$$P_{12}^2 = s = (E+m)^2 - p^2 = \underbrace{E^2 - p^2}_{m^2} + 2mE + m^2$$

$$= 2m(E+m)$$

After $pp \to ppp\bar{p}$

$$P_{\text{Final}}^2 = 16m^2 = s = 2m^2 + 2mE$$

$$14m^2 = 2mE$$

$$\boxed{7m = E}$$