

Linear Oscillations :

Following: Newton - Mechanics

Simple Harmonic Motion

$$F = ma$$
$$-KX = m \ddot{X}$$

$$m \ddot{X} + KX = 0$$

Guess: $X(t) = e^{\lambda t}$

Then $\dot{X} = \lambda e^{\lambda t} = \lambda X$

$$\ddot{X} = \lambda^2 e^{\lambda t} = \lambda^2 X$$

$$m \ddot{X} + KX = 0$$

$$(m \lambda^2 + K) X = 0$$

$$\lambda^2 = -\frac{K}{m}$$

$$\lambda = \pm i \sqrt{\frac{K}{m}} = \pm i \omega_0$$

with $\omega_0 = \sqrt{\frac{K}{m}}$

$$X = C_1 e^{+i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

Oscillating Solution

If we got the sign backwards

$$m \ddot{X} - KX = 0$$

$$m \ddot{X} - KX = 0$$

$$(m \lambda^2 - K) X = 0$$

$$\lambda^2 = +\frac{K}{m}$$

$$\lambda = \pm \sqrt{\frac{K}{m}} = \pm \omega_0$$

$$X = C_1 e^{+\omega_0 t} + C_2 e^{-\omega_0 t}$$

Exponentially Increasing Exponentially damped

Simple Harmonic Motion: Continued

$$X = C_1 e^{+i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

$$v = \dot{X}$$

$$a = \ddot{X}$$

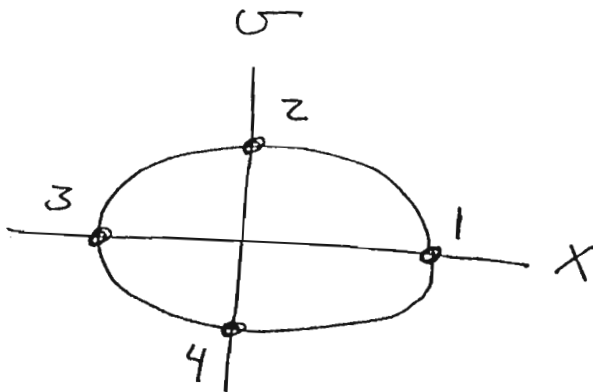
$$E = T + V = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \text{constant}$$

Exercise

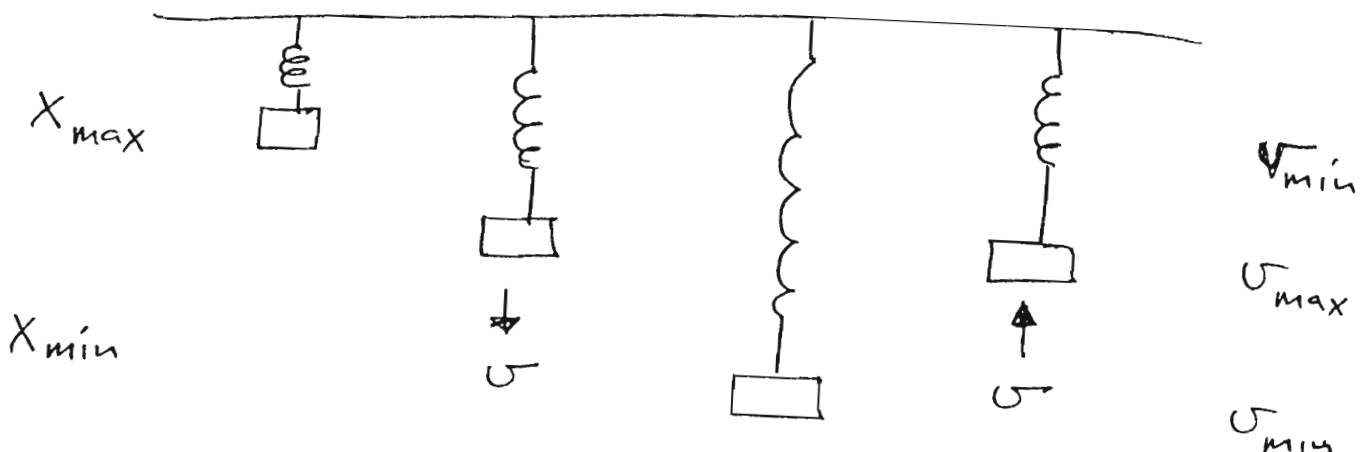
Phase Plot

Equation for circle: $x^2 + y^2 = r^2$

Energy Equation $\approx v^2 + x^2 = E^2$ (Loosely)



Position	X	v
1	+1	0
2	0	+1
3	-1	0
4	0	-1



SHO w/ Damping

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$$m \ddot{x} + b \dot{x} + kx = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0$$

$$\beta = \frac{b}{2m}$$

Guess: $x = e^{\lambda t}$

$$\dot{x} = \lambda x$$

$$\ddot{x} = \lambda^2 x$$

$$\Rightarrow (\lambda^2 + 2\beta\lambda + \omega_0^2) x = 0$$

$$\lambda_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \approx -\beta \pm i\omega_0 \quad \text{for } \beta \ll \omega_0$$

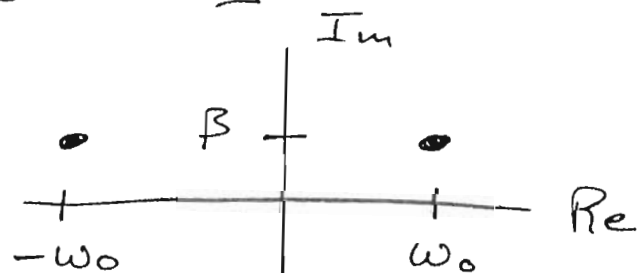
$$x(t) = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t}$$

$$x(t) = e^{-\beta t} \left[c_1 e^{+t\sqrt{\beta^2 - \omega_0^2}} + c_2 e^{-t\sqrt{\beta^2 - \omega_0^2}} \right]$$

Limiting Case $\beta \ll \omega_0$

$$x(t) \approx e^{-\beta t} \left[c_1 e^{+i\omega_0 t} + c_2 e^{-i\omega_0 t} \right]$$

$$\lambda_{\pm} \approx i \left[i\beta \pm \omega_0 \right]$$

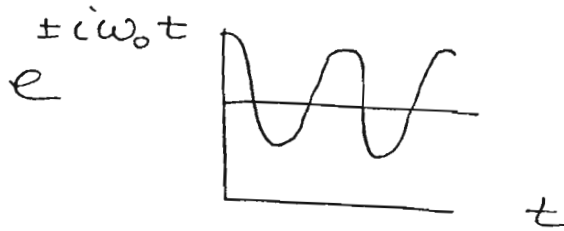


Note: $\beta = \frac{b}{2m}$ controls damping

$\omega_0 = \sqrt{\frac{k}{m}}$ controls oscillations



Note: It is important $\beta \geq 0$



Return to General case:

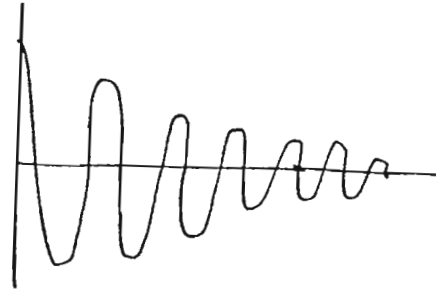
$$X(t) = e^{-\beta t} \left[c_1 e^{+t \sqrt{\beta^2 - \omega_0^2}} + c_2 e^{-t \sqrt{\beta^2 - \omega_0^2}} \right]$$

3 Cases to Consider:

(Let: $\mathcal{R} = |\sqrt{\beta^2 - \omega_0^2}| \geq 0$)

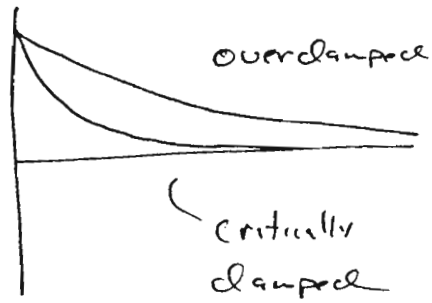
I. Underdamping	II. Critical damping	III. Over-damping
$\omega_0^2 > \beta^2$	$\omega_0^2 = \beta^2$	$\omega_0^2 < \beta^2$
$\sqrt{\beta^2 - \omega_0^2} = i\mathcal{R}$	$\sqrt{\beta^2 - \omega_0^2} = 0$	$\sqrt{\beta^2 - \omega_0^2} = \mathcal{R}$
$\lambda_{\pm} = -\beta \pm i\mathcal{R}$	$\lambda_{\pm} = -\beta + 0$	$\lambda_{\pm} = -\beta \pm \mathcal{R}$
$\lambda_{\pm}$ complex	$\lambda_{\pm}$ Real	$\lambda_{\pm}$ Real

Under damped:



Overdamped

Critically Damped



Exercise

Next

How to solve :

$$m \ddot{x} + b \dot{x} + kx = F(t)$$

Linear Operators

Principle of Superposition:

Let $\mathbb{L}$ be a linear operator.

Examples: $\left(m \frac{d^2}{dt^2} + b \frac{d}{dt} + k \right) x = 0$

or $(m \mathbb{D}^2 + b \mathbb{D} + k) x = 0$

or $\mathbb{D} x = 0$ with $\mathbb{D} = \frac{d}{dt}$

$$\mathbb{L}(x_1 + x_2) = \mathbb{L}(x_1) + \mathbb{L}(x_2)$$

Example:

$$\mathbb{D}(x_1 + x_2) = \frac{d}{dt}(x_1 + x_2) = \frac{dx_1}{dt} + \frac{dx_2}{dt} = \mathbb{D}x_1 + \mathbb{D}x_2$$

Also

$$\mathbb{L}(\alpha x) = \alpha \mathbb{L}(x)$$

For α constant

Example:

$$\mathbb{D}(\alpha x) = \frac{d}{dt}(\alpha x) = \alpha \frac{dx}{dt} = \alpha \mathbb{D}(x)$$

We have already used this idea:

$$\mathcal{L}(x_1) = 0$$

$$(mD^2 + k)x_1 = (mD^2 + k)e^{+i\omega_0 t} = 0$$

$$\mathcal{L}(x_2) = 0$$

$$(mD^2 + k)x_2 = (mD^2 + k)e^{-i\omega_0 t} = 0$$

$$\mathcal{L}(x_1 + x_2) = 0$$

$$(mD^2 + k)(x_1 + x_2) = 0$$

$$(mD^2 + k)\left[c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}\right] = 0$$

New idea:

$$\mathcal{L}(x_G) = 0$$

General Solution

$$\mathcal{L}(x_P) = F(t)$$

Particular Solution

$$\mathcal{L}(x_G + x_P) = F(t)$$

Step 1) Find a particular solution: x_P

Step 2) Add any General solution: x_G

Non-Linear Operators :

Example :

$$X^2 + X + 1 = 0$$

$$\sin(x) = x$$

$$X^{\circ\circ} + (X^{\circ})^2 = 0$$

$$X^2 = a$$

Not linear : $N(x) = x^2$

$$N(x_1 + x_2) = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 \neq N(x_1) + N(x_2)$$

Guess $X = e^{\lambda t}$

For $X^{\circ\circ} + (X^{\circ})^2 = 0$

$$X^{\circ\circ} + (X^{\circ})^2 = 0$$

$$\lambda^2 e^{\lambda t} + (\lambda e^{\lambda t})^2 = 0$$

$$\lambda^2 e^{\lambda t} + \lambda^2 e^{2\lambda t} = 0$$

$$e^{\lambda t} + e^{2\lambda t} = 0$$

$$1 + e^{\lambda t} = 0$$

$$e^{\lambda t} = -1 \quad ???$$

$$\Rightarrow X = e^{\lambda t} = -1$$

Non-sense

Particular Solution: Forced Oscillation:

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$$(m D^2 + b D + k) x = m F e^{i \omega_D t}$$

$$(D^2 + 2\beta D + \omega_0^2) x = F e^{i \omega_D t}$$

$$\begin{cases} \omega_0 = \sqrt{\frac{k}{m}} \\ \beta = \frac{b}{2m} \end{cases}$$

Guess: $x = A e^{i \omega t}$

$$\dot{x} = i \omega x$$

$$\ddot{x} = -\omega^2 x$$

ω w/ no subscript

ω_0 = Natural freq

ω_D = Driving freq

$$\Rightarrow (\omega_0^2 + 2i\beta\omega_D - \omega_D^2) A e^{i \omega_D t} = F e^{i \omega_D t}$$

Now we assume $\omega = \omega_D$

$$A = \frac{F}{(\omega_0^2 + 2i\beta\omega_D - \omega_D^2)} = \frac{-F}{(\omega_D - \omega_+)(\omega_D - \omega_-)}$$

Note:
A is complex

where

$$\omega_{\pm} = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

$$\omega_{\pm} \approx i\beta \pm \omega_0 \quad \text{for } \beta \ll \omega_0$$

$$x = A e^{i \omega_D t} = \frac{-F e^{i \omega_D t}}{(\omega_D - \omega_+)(\omega_D - \omega_-)}$$

Optional: $A = |A| e^{i\delta}$

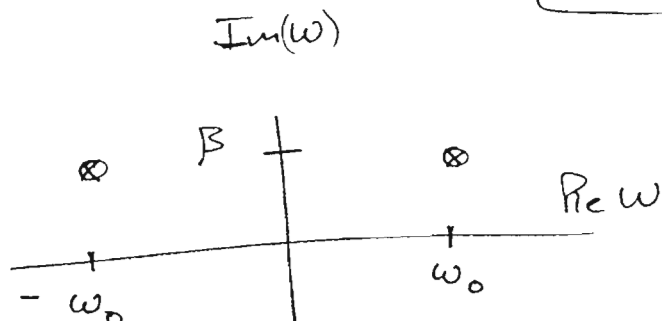
$$x = |A| e^{i(\omega_D t + \delta)}$$

Phase δ :

Resonance: $\delta = +\pi/2 \text{ --- } -\pi/2$

Resonance Structure :

$$X_P = \frac{-F e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)}$$



The particular solution:

$$\omega_{\pm} \approx i\beta \pm \omega_0$$

Recall the general structure:

$$X_G = e^{-\beta t} \left\{ c_1 e^{+t\sqrt{\beta^2 - \omega_0^2}} + c_2 e^{-t\sqrt{\beta^2 - \omega_0^2}} \right\}$$

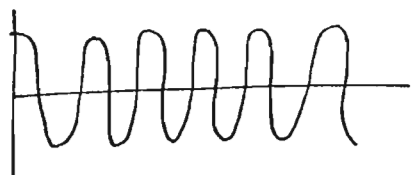
↑ Not present in particular solution

Full Solution

$$X_P + X_G$$

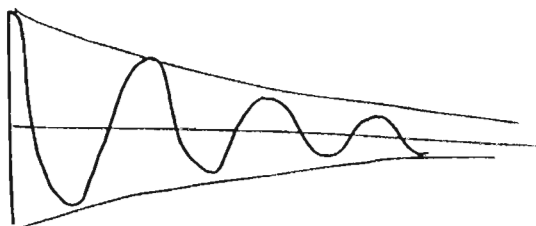
↑ No decay ↑ exponential decay

X_P



- No decay
- Frequency ω determined by driving source

X_G



- Exponential Decay (Transient)
- Frequency $\sim \omega_0$ determined by system - resonance

Forced Oscillations

Instead of $e^{i\omega t}$ = 

try $\theta(t) =$  $= \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

What to do ???

Idea:

$$\mathcal{L}(x_1) = 0$$

$$\mathcal{L}(x_2) = 0$$

$$\mathcal{L}(x_3) = c_3 e^{i\omega_3 t}$$

$$\mathcal{L}(x_4) = c_4 e^{i\omega_4 t}$$

} General
Solutions

} Particular
Solutions

Thus: $\mathcal{L}(x_1 + x_2 + x_3 + x_4) = c_3 e^{i\omega_3 t} + c_4 e^{i\omega_4 t}$

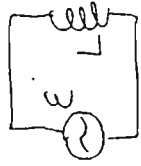
Can we make this
into the function
we need?

In General:

$$\mathcal{L}(c_1 x_1 + c_2 x_2 + \dots) = c_1 F_1 + c_2 F_2 + \dots$$

$$\mathcal{L}\left(\sum_{n=1}^{\infty} c_n x_n\right) = \sum_{n=1}^{\infty} c_n F_n$$

Inductor



$$\mathcal{E} = L \frac{dI}{dt}$$

$$\text{Then } dI = \mathcal{E} dt \frac{1}{L}$$

$$\text{Integrate: } I = \int \mathcal{E} dt \frac{1}{L}$$

$$\text{Let } \mathcal{E}(t) = \mathcal{E}_{\max} \sin(\omega t)$$

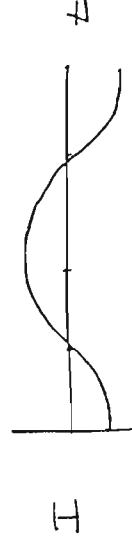
$$I(t) = -\frac{\mathcal{E}_{\max}}{\omega L} \cos(\omega t) = -I_{\max} \cos(\omega t)$$

$$\text{with } I_{\max} = \frac{\mathcal{E}_{\max}}{\omega L} \equiv \frac{\mathcal{E}_{\max}}{X_L}$$

$$\text{with } X_L = \omega L$$



$$\boxed{\mathcal{E} L I}$$



Summary: $RI = V$

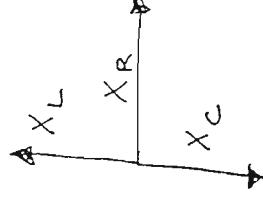
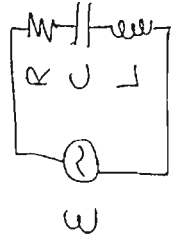
$$XI = \mathcal{E} \Rightarrow I = \frac{\mathcal{E}}{X}$$

Resistor $X_R = R$ I and $\mathcal{E}$ are in phase

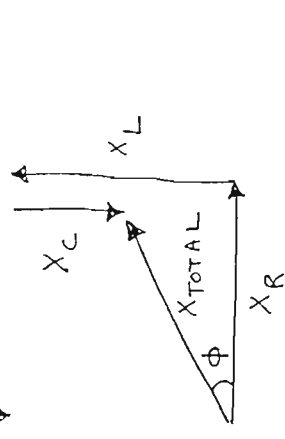
Capacitor $X_C = \frac{1}{\omega C}$ $I \text{ lags } \mathcal{E}$

Inductor $X_L = \omega L$ $\mathcal{E} \text{ lags } I$

Put together RLC Circuit



Add Them up



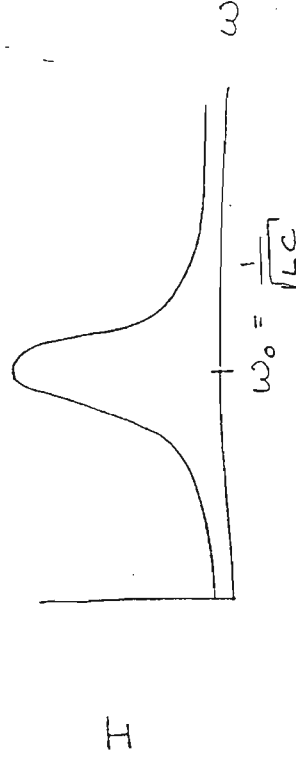
$$Z = X_{\text{TOTAL}} = \sqrt{X_R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$\tan \phi = \frac{X_L - X_C}{X_R}$$

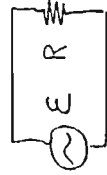
$$RI = V$$

$$ZI = \mathcal{E} \Rightarrow I = \frac{\mathcal{E}}{Z}$$

$$I = \frac{\mathcal{E}}{Z} \Rightarrow I_{\max} = \frac{\mathcal{E}_{\max}}{Z}$$



Resistor



$$RI = V = \mathcal{E}$$

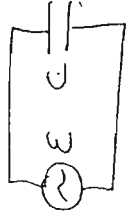
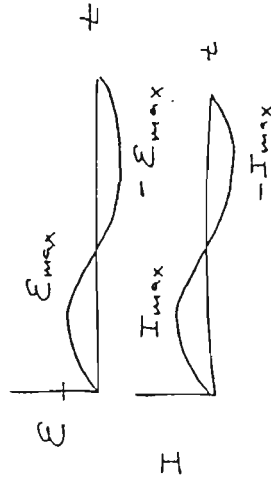
$$I = \frac{\mathcal{E}}{R}$$

$$\text{Let } \mathcal{E}(t) = \mathcal{E}_{\max} \sin(\omega t)$$

$$\text{Then } I(t) = \frac{\mathcal{E}(t)}{R} = \frac{\mathcal{E}_{\max}}{R} \sin(\omega t)$$

$$\text{so } I(t) = I_{\max} \sin(\omega t)$$

$$\text{where } I_{\max} = \mathcal{E}_{\max}/R$$



$$\mathcal{E} = \frac{Q}{C}$$

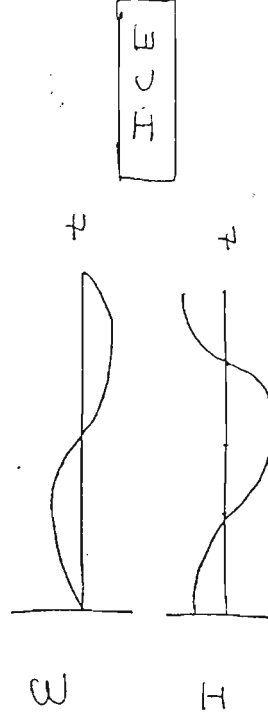
$$\frac{d\mathcal{E}}{dt} = \frac{1}{C} \frac{dQ}{dt} = \frac{1}{C} I$$

$$\text{Let } \mathcal{E}(t) = \mathcal{E}_{\max} \sin(\omega t)$$

$$I(t) = C \frac{d\mathcal{E}(t)}{dt} = C \mathcal{E}_{\max} \omega \cos(\omega t)$$

$$\text{so } I(t) = I_{\max} \cos(\omega t)$$

$$\text{with } I_{\max} = \mathcal{E}_{\max} (\omega C)$$



Note: to make above look like resistor case

$$\text{define } X_C = \frac{1}{\omega C}$$

$$\text{Then for resistor: } I_{\max} = \frac{\mathcal{E}_{\max}}{R}$$

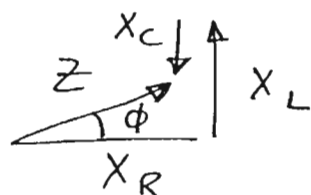
$$\text{For capacitor: } I_{\max} = \frac{\mathcal{E}_{\max}}{X_C} = \mathcal{E}_{\max} \omega C$$

Resonances:

RLC Circuit:

Recall

ELI
ICE



$$RI = V \Rightarrow ZI = \mathcal{E}$$

$$X_R = R$$

$$X_L = \omega L$$

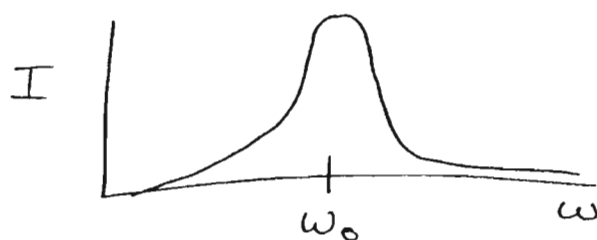
$$X_C = \frac{1}{\omega C}$$

$$Z^2 = R^2 + (X_L - X_C)^2$$

$$\tan \phi = \frac{X_L - X_C}{X_R}$$

Resonant Frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

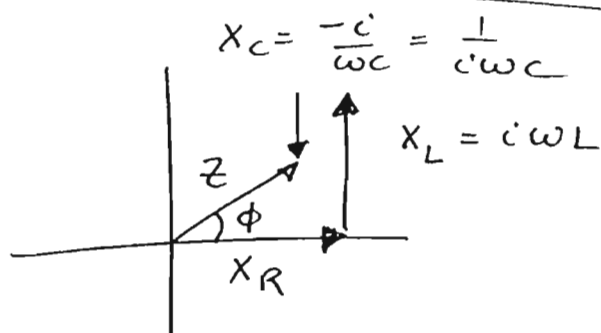


Idea: Use Complex #'s

$$X_R = R$$

$$X_L = i\omega L$$

$$X_C = \frac{1}{i\omega C} = \frac{-i}{\omega C}$$



$$Z = X_R + X_L + X_C$$

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$$I = \mathcal{E}/Z : \text{Peak when } Z=0$$

$$Z=0 \Rightarrow \omega_{\pm} = i\rho \pm \sqrt{\omega_0^2 - \rho^2}$$

$$\begin{cases} \rho = \frac{R}{2L} \\ \omega_0 = \frac{1}{\sqrt{LC}} \end{cases}$$

$$\omega_{\pm} \cong i\rho \pm \omega_0 \quad \text{for small } \rho$$

Complex ω plane

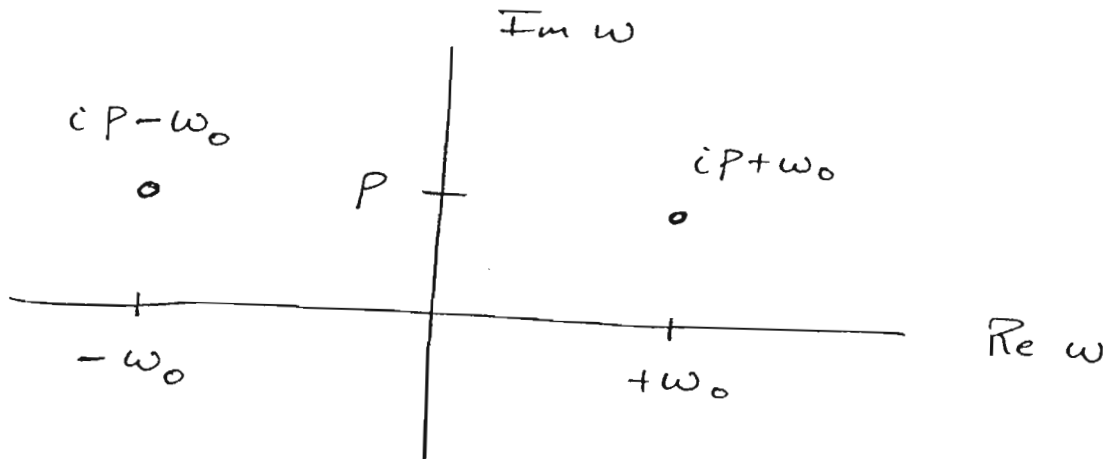
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$$Z = R + i\omega L + \frac{1}{i\omega C}$$

$$Z = \frac{iL}{\omega} (\omega - \omega_+) (\omega - \omega_-)$$

$$\therefore I = \frac{\Sigma}{Z} = \Sigma \frac{\omega}{iL} \frac{1}{(\omega - \omega_+) (\omega - \omega_-)}$$

with $\omega_{\pm} = iP \pm \omega_0$



$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{Resonant Frequency}$$

$$P = \frac{R}{ZL} \quad \text{Resonant width}$$

