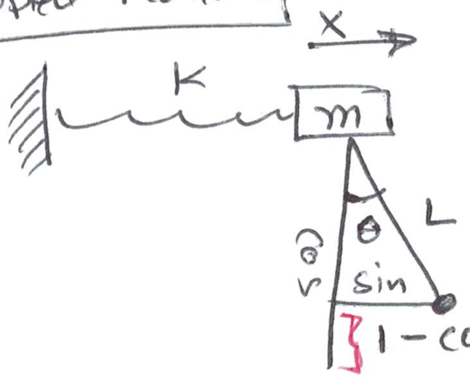
Coupled Motion



$\sin\theta \sim \theta$

$1-\cos\theta \cong \dfrac{\theta^2}{2}$ ← $\propto$ height of M

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \underbrace{(\dot{x} + L\dot{\theta})}_{\text{horizontal velocity of } M}{}^2$$

For small angles, we ignore vertical velocity $\sim \dot{\theta}^4$

$$V = \frac{1}{2} K x^2 + M g \underbrace{L \frac{\theta^2}{2}}_{\text{height of } M}$$

$$\mathcal{L} = T - V$$

$\boxed{10 \text{ pts}}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \qquad (m+M)\ddot{x} + LM\ddot{\theta} + Kx = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad LM\ddot{x} + L^2 M\ddot{\theta} + MgL\theta = 0$$

$$T_{matrix} = \begin{pmatrix} m+M & LM \\ LM & L^2 M \end{pmatrix} \quad V_{matrix} = \begin{pmatrix} K & 0 \\ 0 & gLM \end{pmatrix}$$

$\boxed{5 \text{pts}}$

Eq of Motion $\Rightarrow$ $V_{mat} - T_{mat}\,\omega^2 = 0$

$$\begin{pmatrix} K-(m+M)\omega^2 & -LM\omega^2 \\ -LM\omega^2 & gLM - L^2 M\omega^2 \end{pmatrix} = 0$$

Set $m = M = L = g = 1$ and $K = 2$

$$\Rightarrow \begin{pmatrix} 2-2\omega^2 & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{pmatrix} = 0$$

$$\text{Det} = \omega^4 - 4\omega^2 + 2 = 0$$

$\boxed{\omega^2 = 2 \pm \sqrt{2}}$ Eigen frequencies

5pts

Find Normal Modes

$\omega^2 = 2 + \sqrt{2}$

$$V_{mat} - T_{mat}\,\omega^2 \longrightarrow \begin{pmatrix} \cdots & \cdots \\ -2-\sqrt{2} & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow (-2-\sqrt{2})\,a + (-1-\sqrt{2})\,b = 0$$

$$\Rightarrow b = -a \frac{(-2-\sqrt{2})}{(-1-\sqrt{2})} \frac{(-1+\sqrt{2})}{(-1+\sqrt{2})} = -\sqrt{2}\,a$$

$\therefore$ Eigenvector #1 $= v_1 = (1, -\sqrt{2})$

$\omega_1 = 2 + \sqrt{2} \cong 3.4$

$\omega_2 = 2 - \sqrt{2}$

$$V - T\omega^2 \longrightarrow \begin{pmatrix} \circ\circ\circ & \circ\circ\circ \\ -2+\sqrt{2} & -1+\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

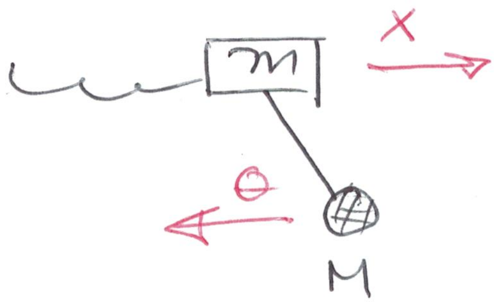
$$\Rightarrow b = -a \frac{(-2+\sqrt{2})}{(-1+\sqrt{2})} \frac{(-1-\sqrt{2})}{(-1-\sqrt{2})} = +\sqrt{2}\,a$$

Eigenvector #2 $= v_2 = (1, +\sqrt{2})$

$\omega_2 = 2 - \sqrt{2} \cong 0.6$

$\boxed{\text{5pts}}$

Mode #1: $\omega_1 \sim 3.4 > \omega_2 \sim 0.6$

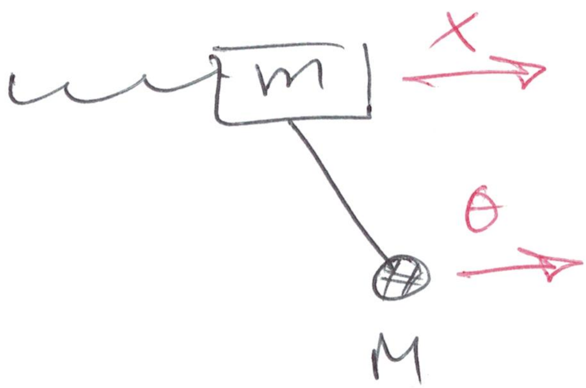


$v_1 = (1, -\sqrt{2})$

Anti-symmetric

- Motion is in opposite directions
- Frequency is LARGE

Mode #2 $\omega_2 \sim 0.6 < \omega_1 \sim 3.4$



$v_2 = (1, +\sqrt{2})$

Symmetric

- Motion is in same direction
- Frequency is small

$\boxed{5\text{pts}}$

Rubric 30 pts

10 pts: correct T, V, L including small angle approximation for θ. $\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

5 pts correct T_{matrix} V_{matrix} from Eq of motion.

5 pts correct $\omega^2_{1,2}$ from $(\overleftrightarrow{V} - \overleftrightarrow{T}\omega^2) = 0$

5 pts correct Eigen Vectors υ_1, υ_2

5 pts Interpretation of $\upsilon_{1,2}$ $\omega^2_{1,2}$

- Identification of symmetric / anti-sym. modes
- Note that $\omega_{sym} < \omega_{Asym}$

Note: They can get full credit on final part even if they get details of $\upsilon_{1,2}$ and $\omega_{1,2}$ incorrect