Fourier Transforms

3/23/97

Preliminaries:

Take $n, m \in$ Integers

$$I = \int_0^{2\pi} dx \; e^{inx} \cdot \left(e^{imx}\right)^* = \int_0^{2\pi} dx \; e^{i(n-m)x}$$

$$= \left.\frac{e^{i(n-m)x}}{i(n-m)}\right]_0^{2\pi} = \frac{e^{i(n-m)2\pi} - e^{0}}{i(n-m)}$$

$$= 0 \quad \text{if } n \neq m$$

If $n = m$:

$$I = \int_0^{2\pi} dx \cdot 1 = 2\pi$$

$$\therefore \int_0^{2\pi} dx \; e^{inx} \left(e^{imx}\right)^* = \begin{cases} 0 & \text{if } n \neq m \\ 2\pi & \text{if } n = m \end{cases}$$

$$\equiv 2\pi \, \delta_{n,m}$$

$$\boxed{\int_0^{2\pi} dx \; e^{inx} \left(e^{imx}\right)^* = 2\pi \, \delta_{n,m}}$$

Orthogonality !!!

Likewise:

$$\int_0^{2\pi} dx \quad \sin(nx)\,\sin(mx) = \begin{cases} \pi\,\delta_{n,m} \\ 0 \quad \text{for } n=m=0 \end{cases}$$

$$\int_0^{2\pi} dx \quad \cos(nx)\,\cos(mx) = \begin{cases} \pi\,\delta_{n,m} \\ 2\pi \quad \text{for } n=m=0 \end{cases}$$

$$\int_0^{2\pi} dx \quad \sin(nx)\,\cos(mx) = 0$$

Short cut: $\sin^2 + \cos^2 = 1$

suggests $\overline{\sin}^2 = \overline{\cos}^2 = \frac{1}{2}$

$$\therefore \quad \int_0^{2\pi} dx\, \sin^2 = \int_0^{2\pi} dx \cdot \frac{1}{2} = \frac{1}{2} \cdot 2\pi = \pi$$

Likewise: $\int_0^{2\pi} dx\, \cos^2 = \pi$

Fourier Series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

Goal: Find c_n:

$$\int_0^{2\pi} dx \; e^{-imx} F(x) = \int_0^{2\pi} dx \sum_{n=-\infty}^{\infty} c_n \, e^{i(n-m)x}$$

$$= \sum_n c_n \underbrace{\int_0^{2\pi} dx \; e^{i(n-m)x}}_{2\pi \, \delta_{n,m}} = 2\pi \, c_m$$

$$\therefore \quad \boxed{c_m = \frac{1}{2\pi} \int_0^{2\pi} dx \; e^{-imx} \, F(x)}$$

Similar Derivation:

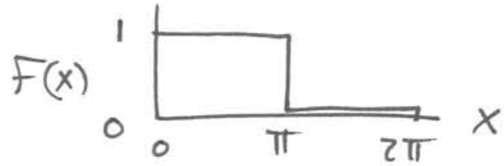
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} dx \; \cos(nx) \; F(x)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} dx \; \sin(nx) \; F(x)$$

Note: $c_n = \frac{1}{2}(a_n - i b_n) \equiv \frac{1}{2\pi} \int_0^{2\pi} dx \; e^{-inx} F(x)$

Example #1 – A



C.F., P. 819

$$F(x) = \begin{cases} 1; & x \in [0,\pi] \\ 0; & x \in [\pi, 2\pi] \end{cases} \qquad a_0 = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} dx \, \cos(nx) \, F(x) \equiv \frac{1}{\pi}\int_0^{\pi} dx \, \cos(nx) \cdot 1 = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} dx \, \sin(nx) \, F(x) = \frac{1}{\pi}\int_0^{\pi} dx \, \sin(nx) \cdot 1$$

$$= \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-(-1)^n + 1}{n}\right] = \begin{cases} \frac{2}{n\pi} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{a_0}{2} + \sum_{n=1,3,5\ldots} \frac{2}{n\pi} \sin(nx)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \ldots\right]$$

Example #1-B

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} dx \left(e^{inx}\right)^* F(x) = \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-inx} F(x)$$

$$= \frac{1}{2\pi}\int_0^{\pi} dx\, e^{-inx} \cdot 1 = \frac{1}{2\pi}\left[\frac{e^{-inx}}{inx}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi} - 1}{-in}\right] = \frac{1}{2\pi}\left[\frac{(-1)^n - 1}{-in}\right] =$$

$$C_n = \begin{cases} \dfrac{-i}{n\pi} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} dx \cdot 1 \cdot F(x) = \frac{1}{2}$$

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \frac{1}{2} + \sum_{\substack{n=1,3,5 \\ -1,-3,-5}} \left(\frac{-i}{n\pi}\right) e^{inx}$$

$$= \frac{1}{2} + \sum_{n=1,3,5} \frac{-i\cos(nx) + \sin(nx)}{n\pi}$$

$$+ \sum_{+n=1,3,5} \frac{-i\cos(-nx) + \sin(-nx)}{-n\pi}$$

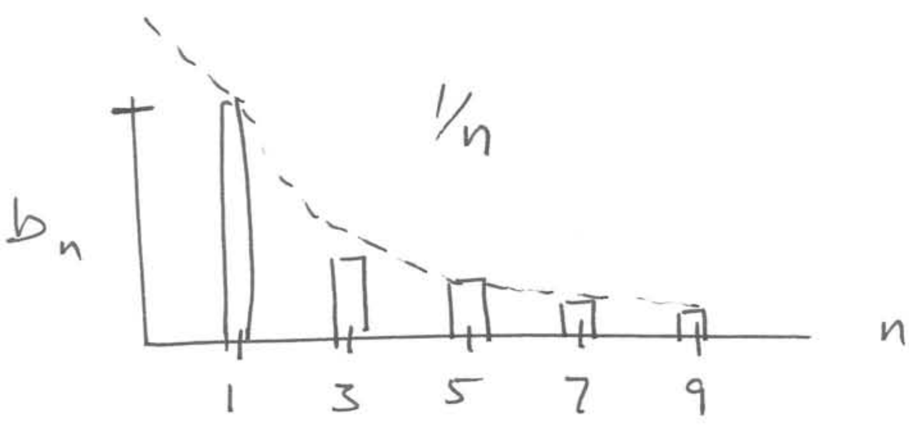
$$= \frac{1}{2} + \sum_{n=1,3,5} \frac{\sin(nx)}{n\pi} + \sum_{+n=1,3,5} \frac{-\sin(nx)}{-n}$$

$$= \frac{1}{2} + \sum_{n=1,3,5} \left(\frac{2}{n\pi}\right) \sin(nx)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right]$$

Same as Example #1-A

$$b_n = \frac{2}{\pi}\frac{1}{n} \quad \text{For } n = \text{odd}$$



Continuous Fourier Transform:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; g(\omega) \; e^{-i\omega x}$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; F(x) \; e^{+i\omega x}$$

Check:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, F(x') \, e^{i\omega x'} \right\} e^{-i\omega x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \; F(x') \; \underbrace{\int_{-\infty}^{\infty} d\omega \; e^{i\omega(x'-x)}}_{2\pi \, \delta(x'-x)}$$

$$= 1 \cdot \int_{-\infty}^{\infty} dx' \; F(x') \; \delta(x'-x)$$

$$F(x) = F(x)$$

Note:

$$2\pi \, \delta(x-x') = \int d\omega \; e^{i\omega(x-x')}$$

Greens Functions:

Hard Problem: $\mathbb{D}\, F(x) = \phi(x)$

Easier Problem: $\mathbb{D}\, G(x) = \delta(x)$

or $\mathbb{D}\, G(x-x') = \delta(x-x')$

If we solve easier problem, we can solve hard problem. because:

$$F(x) = \int_{-\infty}^{\infty} G(x-x')\, \phi(x')\, dx'$$

Check:

$$\mathbb{D}\, F(x) = \int_{-\infty}^{\infty} \underbrace{\mathbb{D}\, G(x-x')}_{\delta(x-x')}\, \phi(x')\, dx'$$

$$= \int_{-\infty}^{\infty} dx'\, \delta(x-x')\, \phi(x')$$

$$\mathbb{D}\, F(x) = \phi(x)$$

QED.

Now, how to solve easy problem:

Use Fourier transforms.
Turns differential eq $\Rightarrow$ algebraic eq.

$\boxed{\text{Ex.}}$ $$(a\, D^2 + bD + c)\, F(x) = \delta(x)$$

use: $$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, g(\omega)\, e^{-i\omega x}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{\pm i\omega x}$$ $\left\{ \text{either sign works} \right.$

$\therefore$

$$(aD^2 + bD + c) \frac{1}{\sqrt{2\pi}} \int d\omega\, g(\omega)\, e^{-i\omega x}$$

$$= \frac{1}{\sqrt{2\pi}} \int d\omega\, g(\omega) \left[-\omega^2 a - i\omega b + c\right] e^{-i\omega x}$$

$$\equiv \frac{1}{2\pi} \int d\omega \quad e^{-i\omega x}$$

$\therefore \Rightarrow$ $$g(\omega)\left[-\omega^2 a - i\omega b + c\right] = \frac{1}{\sqrt{2\pi}}$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\left[-\omega^2 a - i\omega b + c\right]}$$

Full solution:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int d\omega \quad g(\omega) \quad e^{-i\omega x}$$

$$= \frac{1}{2\pi} \int d\omega \quad \frac{e^{-i\omega x}}{\left[-\omega^2 a - i\omega b + c\right]}$$

Example Schrodinger Equation:

Guess answer is wave: $\psi(x) = e^{+iPX/\hbar}$

$$\nabla \psi = \frac{iP}{\hbar}\psi \Rightarrow P = -i\hbar\nabla$$

$$\nabla^2 \psi = -\frac{P^2}{\hbar^2}\psi \Rightarrow P^2 = -\hbar^2\nabla^2$$

$$E = \frac{1}{2}mv^2 = \frac{P^2}{2m} \quad \text{with} \quad P = mv$$

$\therefore$

$$E = \frac{P^2}{2m} = \frac{-\hbar^2\nabla^2}{2m}$$

or

$$\boxed{E\psi = \frac{-\hbar^2\nabla^2}{2m}\psi}$$

Check: $\psi = e^{+iPX/\hbar}$

$$\nabla^2\psi = \frac{-P^2}{\hbar^2}\psi$$

$\therefore$

$$E\psi = \frac{-\hbar^2}{2m}\nabla^2\psi = \frac{-\hbar^2}{2m}\frac{-P^2}{\hbar^2}\psi = \frac{+P^2}{2m}\psi$$

$$\Rightarrow \quad E\psi = \frac{P^2}{2m}\psi \qquad \text{Check.}$$

Momentum Representation

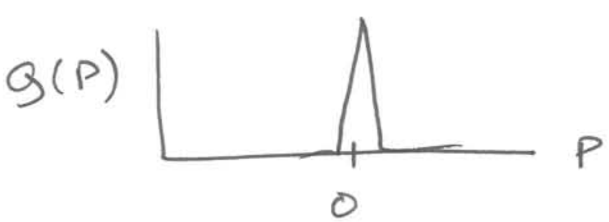
$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dP \; g(P) \; e^{+iPx/\hbar}$$

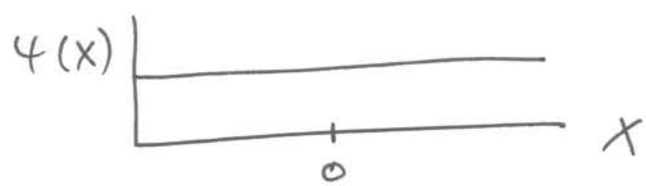
$$g(P) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \; \psi(x) \; e^{-iPx/\hbar}$$

Example: Let $g(P) = \delta(P)$

$$\therefore \quad \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dP \; \delta(P) \; e^{iPx/\hbar}$$

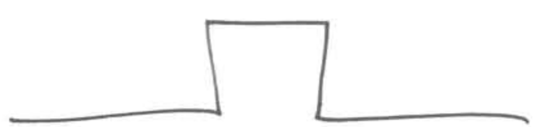
$$= \frac{1}{\sqrt{2\pi\hbar}} \cdot 1 \qquad = \text{const.}$$



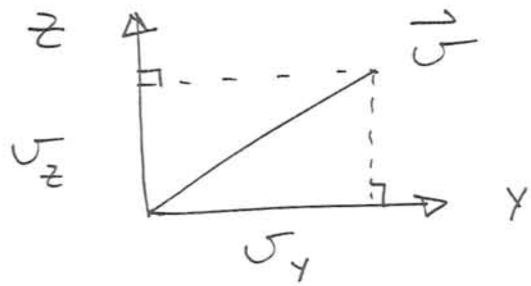


What do you expect for:

a) $g(P) = e^{-P^2x^2/\hbar^2}$

b) $g(P) =$ 

Fourier Transforms: (An Analogy)



$$v_y = \vec{y} \circ \vec{v} = \langle y | v \rangle$$

$$v_z = \vec{z} \circ \vec{v} = \langle z | v \rangle$$

$$\vec{v} = \hat{y}\, v_y + \hat{z}\, v_z$$

$$|v\rangle = |y\rangle\langle y|v\rangle + |z\rangle\langle z|v\rangle$$

"divide" by $|v\rangle$

$$\overleftrightarrow{\mathbb{1}} = |y\rangle\langle y| + |z\rangle\langle z| \equiv \sum_i |x_i\rangle\langle x_i|$$

$\vec{x}_i$ vector

projection on $\vec{x}_i$

Works for any vector

$$|w\rangle = \left(\sum_i |x_i\rangle\langle x_i|\right)|w\rangle$$

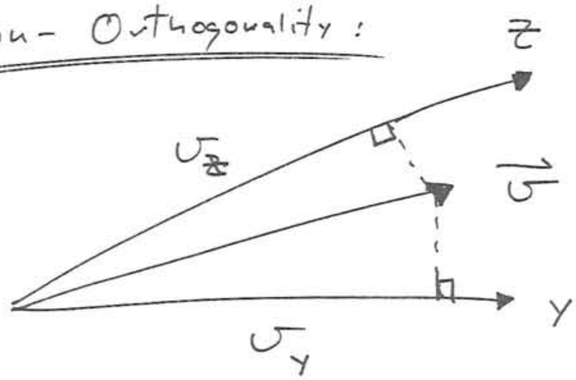
$$= \sum_i \underbrace{|x_i\rangle}_{\text{basis vector}} \underbrace{\langle x_i | w\rangle}_{\text{projection coefficient, a number}}$$

Orthogonality:

$$x_i \circ x_j = \delta_{ij}$$

$$\langle x_i | x_j \rangle = \delta_{ij}$$

Non-Orthogonality:



$$v_y = y \circ v = \langle y | v \rangle$$

$$v_z = z \circ v = \langle z | v \rangle$$

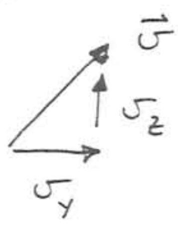
Note: $\vec{v} \neq v_y \, \hat{y} + v_z \, \hat{z}$

$$\neq |y\rangle\langle y|v\rangle + |z\rangle\langle z|v\rangle$$

$$\neq \sum_i |x_i\rangle\langle x_i|v\rangle$$

Completeness:

$$\overleftrightarrow{\mathbb{1}} = |y\rangle\langle y| + |z\rangle\langle z| = \sum_i |x_i\rangle\langle x_i|$$



$$\vec{v} = \hat{y} \, v_y + \hat{z} \, v_z$$

$$|v\rangle = |y\rangle\langle y|v\rangle + |z\rangle\langle z|v\rangle$$

This would not work if we left out - say - y

$$\overleftrightarrow{\mathbb{1}} \neq 0 + |z\rangle\langle z| \qquad \vec{v} \neq 0 + \hat{z} \, v_z$$

Change from vectors $\vec{v} = |v\rangle$ to functions $F(x) = |F\rangle$

$$F(x) = |F\rangle = 2\sin(2x) + 3\sin(3x)$$

$$|F\rangle \equiv 2\,|X_2\rangle + 3\,|X_3\rangle$$

$$|X_2\rangle = \sin(2x)$$

$$|X_3\rangle = \sin(3x)$$

$$\langle a|b\rangle = \frac{1}{\pi}\int_0^{2\pi} dx \; a^* b$$

$$\therefore \quad |F\rangle = \left(|x_2\rangle\langle x_2| + |x_3\rangle\langle x_3|\right)|F\rangle$$

$$= |x_2\rangle\langle x_2|F\rangle + |x_3\rangle\langle x_3|F\rangle$$

$$\langle x_2|F\rangle = \frac{1}{\pi}\int_0^{2\pi} dx \; \sin(2x)\left[2\sin(2x) + 3\sin(3x)\right]$$

$$= \frac{1}{\pi}\int_0^{2\pi} dx \left[2\underbrace{\sin^2(2x)}_{1/2} + 3\underbrace{\sin(2x)\sin(3x)}_{0}\right]$$

$$= \frac{1}{\pi}\cdot 2\pi \cdot 2 \cdot \frac{1}{2} = 2$$

Likewise:

$$\langle x_3|F\rangle = \frac{1}{\pi}\cdot 2\pi\cdot 3\cdot\frac{1}{2} = 3$$

$$\therefore \quad |F\rangle = |x_2\rangle\langle x_2|F\rangle + |x_3\rangle\langle x_3|F\rangle$$

$$= \sin(2x)\cdot 2 + \sin(3x)\cdot 3$$

A perfect match.

(designed that way)

The Bra Ket Notation

$\langle$ Bra$|$ $\quad$ $|$Ket$\rangle$

$\langle v| = (y \; z) \qquad |v\rangle = \begin{pmatrix} y \\ z \end{pmatrix}$ $\qquad$ Vectors

$\langle v|v\rangle = v \circ v = (y \; z) \circ \begin{pmatrix} y \\ z \end{pmatrix} = y^2 + z^2$ $\qquad$ scalar number

Unit Vectors: $\langle y| = (1 \; 0) \qquad |y\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{y} \quad \hat{z}$ $\qquad \langle z| = (0 \; 1) \qquad |z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle y|y\rangle = 1 = \langle z|z\rangle$ and $\langle y \, z\rangle = 0 = \langle z \, y\rangle$

$|v\rangle\langle v| = \begin{pmatrix} y \\ z \end{pmatrix}(y \; z) = \begin{pmatrix} yy & yz \\ zy & zz \end{pmatrix}$ $\qquad$ a matrix

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Vector: $\langle v|$ or $|v\rangle$ $\qquad$ Number $\langle v|v\rangle$

Matrix $|v\rangle\langle v|$

Unit Matrix $= \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |y\rangle\langle y| + |z\rangle\langle z| = \sum_i |x_i\rangle\langle x_i|$

Project a vector:

$\langle v| = \langle v| \mathbb{1} = \langle v| \{ |y\rangle\langle y| + |z\rangle\langle z| \}$

$= \langle v|y\rangle\langle y| + \langle v|z\rangle\langle z|$

$= v_y \, \hat{y} + v_z \, \hat{z}$

Project v on y $\qquad$ Project v on z