2 Masses on a Spring

$$T = \frac{1}{2} m \left(\dot{X}_1^2 + \dot{X}_2^2 \right) \qquad V = \frac{1}{2} K (X_1 - X_2)^2$$

$$L = T - V = \frac{m}{2}\left(\dot{X}_1^2 + \dot{X}_2^2 \right) - \frac{K}{2}(X_1 - X_2)^2$$

$$\left[X_1^2 + X_2^2 - 2X_1X_2 \right]$$

$$\frac{\partial L}{\partial X_1} = -\frac{K}{2}\left[2X_1 - 2X_2 \right] \qquad \frac{\partial L}{\partial X_2} = -\frac{K}{2}\left[2X_2 - 2X_1 \right]$$

$$\frac{\partial L}{\partial \dot{X}_1} = m\dot{X}_1 \qquad \frac{\partial L}{\partial \dot{X}_2} = m\dot{X}_2$$

$$(1) \Rightarrow \quad m\ddot{X}_1 + K\left[X_1 - X_2 \right] = 0$$

$$(2) \Rightarrow \quad m\ddot{X}_2 + K\left[-X_1 + X_2 \right] = 0$$

$$\overleftrightarrow{M}\ddot{X} + \overleftrightarrow{K} X = 0 \qquad \ddot{X} = \begin{pmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{pmatrix} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\overleftrightarrow{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \overleftrightarrow{K} = K\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Solve $|K - \omega^2 M| = 0$

$$K\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} - \omega^2\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix} = \begin{pmatrix}K - \omega^2 m & -K\\ -K & K - \omega^2 m\end{pmatrix}$$

$$(K - \omega^2 m)^2 - K^2 = 0$$

$$\omega^2 m\,[\omega^2 m - 2K] = 0 \Rightarrow \boxed{\omega^2 = 0,\ \frac{2K}{m}}$$

Find Eigen Modes $\qquad \omega_1 = 0$

$$(K - \omega_1^2 M)\,v_1 = 0 \qquad v_1 = \begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

$$K\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = 0 \Rightarrow \begin{aligned} a_1 - a_2 &= 0\\ -a_1 + a_2 &= 0 \end{aligned}$$

$$a_1 = a_2 \qquad \boxed{v = \begin{pmatrix}1\\ 1\end{pmatrix}}$$

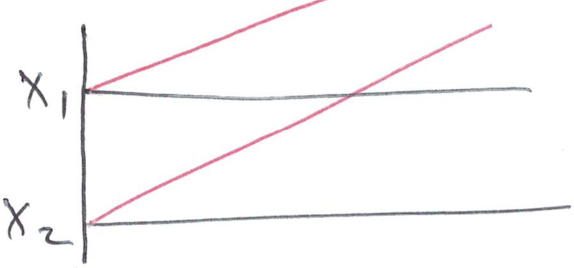
$$\omega_2^2 = \frac{2K}{m}$$

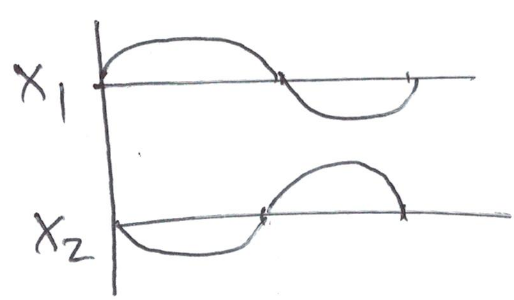
$$(K - \omega_2^2 M) = K\begin{pmatrix}1-2 & -1\\ -1 & 1-2\end{pmatrix} = -K\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = 0 \Rightarrow a_1 + a_2 = 0$$

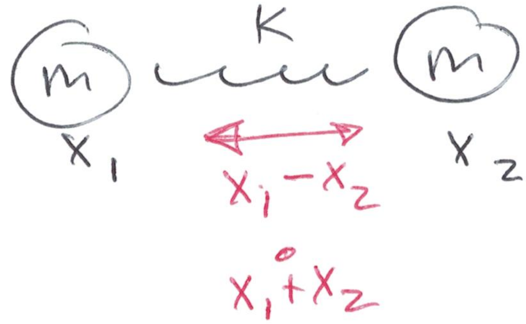
$$v_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$$

$v_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$ $\qquad\qquad$ $v_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$





2 Masses on Spring c̄ Alternate



Let $\boxed{\begin{aligned} Y_1 &= \frac{x_1 + x_2}{2} \\ Y_2 &= \frac{x_1 - x_2}{2} \end{aligned}}$

Then $x_1 = (Y_1 + Y_2)$ $\quad x_2 = (Y_1 - Y_2)$

$L = T - V \qquad T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) = m\left(\dot{Y}_1^2 + \dot{Y}_2^2\right)$

cross terms vanish [check this]

$V = \frac{k}{2}(x_1 - x_2)^2 = 2kY_2^2$

$L = m\left(\dot{Y}_1^2 + \dot{Y}_2^2\right) - 2kY_2^2$

$$\begin{aligned} ① &\Rightarrow \boxed{2m\ddot{Y}_1 + 0 = 0} \\ ② &\Rightarrow \boxed{2m\ddot{Y}_2 + 4kY_2 = 0} \end{aligned}$$

① $\ddot{Y}_1 = 0 \Rightarrow Y(t) = Y_0 + v_{0Y}\, t$

② $m\ddot{Y}_2 + 2k\, Y_2 = 0 \Rightarrow Y_2 = A e^{\pm i\omega t}$

$\omega^2 = \frac{2k}{m}$

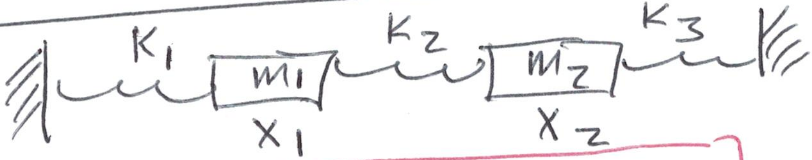
$\boxed{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ in Y basis

$\boxed{v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

In X-basis

$\boxed{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Two Masses + 3 Springs



Simplify $K_1 = K_2 = K_3 = K$

$m_1 = m_2 = m$

(11.2) ① $m\ddot{x}_1 = -2kx_1 + kx_2$

② $m\ddot{x}_2 = kx_1 - 2kx_2$

Let: $y_1 = \frac{1}{2}(x_1 + x_2)$ $\quad y_2 = \frac{1}{2}(x_1 - x_2)$

$x_1 = (y_1 + y_2)$ $\quad x_2 = (y_1 - y_2)$

① $m(\ddot{y}_1 + \ddot{y}_2) = -k(y_1 + 3y_2)$

② $m(\ddot{y}_1 - \ddot{y}_2) = -k(y_1 - 3y_2)$

① + ② $\Rightarrow$ $2m\ddot{y}_1 = -2ky_1$

① − ② $\Rightarrow$ $2m\ddot{y}_2 = -2k(3y_2)$

$$\boxed{m\ddot{y}_1 + ky_1 = 0}$$

$$\boxed{m\ddot{y}_2 + 3ky_2 = 0}$$

$y_1 = A_1 e^{\pm i\omega_1 t}$

$y_2 = A_2 e^{\pm i\omega_2 t}$

$\omega_1^2 = \frac{k}{m}$ $\quad \omega_2^2 = \frac{3k}{m}$

$$x_1(t) = y_1 + y_2$$

$$x_2(t) = y_1 - y_2$$

See Sec 11.2