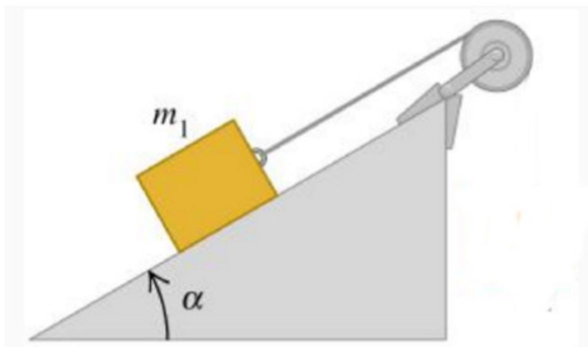
Classical Mechanics Placement Exam

Instructions

The exam consists of 3 questionss. All problems will be graded. You have two hours to work on the solutions. This is a **closed-book exam**.

Problem 1 [10 Points]



Consider an Atwood machine with mass m_1 on a frictionless ramp with a **MASSIVE** pulley of mass m_2 and moment of inertia of $I = (2/5)m_2r^2$. The string is wound around the pulley and does not slip. For coordinates, measure the distance of m_1 along the ramp to be x, and the rotation of the pulley to be θ.

1. (5 Points) Compute the Lagrangian L, and obtain the associated equations of motion in terms of $\{x, x', \theta, \theta'\}$ using a Lagrange multiplier λ.
2. (5 Points) Find the acceleration of the m_1. Also solve the Lagrange multiplier λ and compare this to the tension T in the string.

Problem 2 [20 Points]

Suppose you have a solid cylinder (m = 0.5 kg) attached to a horizontal spring (k = 1.5 N/m), and the cylinder rolls without slipping along a horizontal surface. If the cylinder is released from a point 15 cm from the system's equilibrium position, then:

1. (10 Points) What are the translational and rotational kinetic energies of the cylinder when it is passing through its equilibrium position?
2. (10 Points) What is the frequency of the simple harmonic motion due to the spring?

Problem 3 [20 Points]

1. (2 Points) A muon has a lifetime of 2.2μs. The muon is observed by a person at rest on the surface of the Earth to be traveling at approximately the speed of light. How far (from the perspective of the Earth observer) will the muon typically travel before it decays? *For this question, ignore relativistic effects on the muon.*

2. (3 Points) Now consider relativistic effects on the observed lifetime of the muon. To make a muon collider, let us suppose the muon must typically travel 10km. Compute the relativistic γ factor needed to achieve this goal. (That is, we want the muon to travel 10km during one lifetime of 2.2μs, observed from the perspective of a person at rest on the surface of the Earth, in the so-called "laboratory frame.")

3. (1 Points) Use part b) to now compute β.

4. (2 Points) An observer is at rest on an asteroid. From the perspective of that observer, two space ships approach each other each traveling at $v = 0.7c$. When they pass, what is their relative speed from the perspective of an observer on either ship? *(Hint, the answer is NOT $1.4c$.)*

5. (6 Points) The Z boson is a neutral particle with a mass of 90 GeV/c^2. (For reference, in these units the proton mass is 1 GeV/c^2). If we create a Z boson in the process: $pp \to ppZ$, what is the minimum energy required of each initial-state proton if we collide them in the center-of-mass frame?

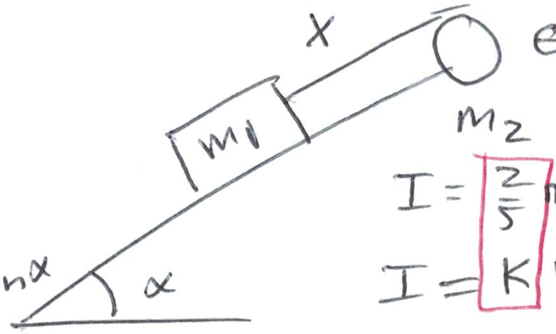
6. (6 Points) Repeat part e), but now do it in the frame where one proton is at rest; what is the minimum energy of the initial-state moving proton required to create the Z boson?

For further reference:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\
\beta &= \frac{v}{c} \\
\beta_{12} &= \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}
\end{aligned}$$

Problem #1

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$



$$I = \boxed{\frac{2}{5}} m_2 r^2$$

$$I = \boxed{K} m_2 r^2$$

$$V = mgh = -m_1 g x \sin\alpha$$

$$L = T - V = \frac{m_1}{2}\dot{x}^2 + \frac{I}{2}\dot{\theta}^2 + m_1 g x \sin\alpha$$

$$\frac{\partial L}{\partial x} = mg\sin\alpha \qquad \frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} \qquad \frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

Constraint: $x = r\theta \Rightarrow f = x - r\theta = 0$

$$\frac{\partial f}{\partial x} = +1 \qquad \frac{\partial f}{\partial \theta} = -r \qquad \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \lambda \frac{\partial f}{\partial q}}$$

$\boxed{x}$ $\quad m_1\ddot{x} - m_1 g\sin\alpha = \lambda$

$\boxed{\theta}$ $\quad I\ddot{\theta} - 0 = -r\lambda \Rightarrow K m_2 r^2 \frac{\ddot{x}}{r} = -r\lambda$

$\boxed{f}$ $\quad f \Rightarrow \ddot{x} = r\ddot{\theta}$ $\qquad K m_2 \ddot{x} = -\lambda$

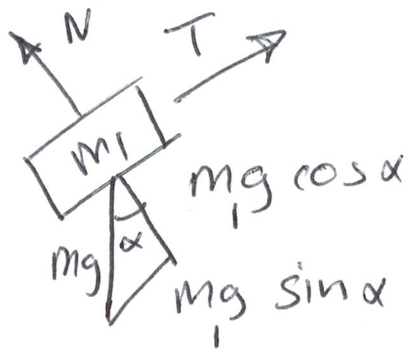
$$x \Rightarrow m_1\ddot{x} - m_1 g\sin\alpha = \lambda = -K m_2 \ddot{x}$$

$$\ddot{x}(m_1 + K m_2) = m_1 g \sin\alpha$$

$$\ddot{x} = \frac{m_1 g\sin\alpha}{(m_1 + K m_2)} \qquad K = \frac{2}{5}$$

$$\lambda = -K m_2 \ddot{x} = -\frac{K m_1 m_2 g\sin\alpha}{(m_1 + K m_2)}$$

Force Method



$$F = ma$$

① $m_1 g\sin\alpha - T = m_1 a$

$$\tau = I\alpha$$

$$rT = I\alpha \qquad \alpha = \frac{a}{r}$$

$$rT = K m_2 r^2 \frac{a}{r}$$

② $T = K m_2 a$

① ② $\rightarrow$ $m_1 g\sin\alpha - K m_2 a = m_1 a$

$$\boxed{\frac{m_1 g\sin\alpha}{m_1 + K m_2} = a}$$

$$T = K m_2 a = K \frac{m_1 m_2 g\sin\alpha}{m_1 + K m_2}$$

Problem #2

$X=0$ is Equilibrium

$$E_{before} = E_{After}$$

$U = T$ Kinetic Energy

$\gamma = \frac{1}{2}$ my notation

$$\frac{1}{2}KX^2 = \frac{1}{2}m v^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m v^2 + \frac{1}{2}\gamma m r^2 \frac{v^2}{r^2}$$

$$\frac{1}{2}KX^2 = \frac{1}{2}m v^2 (1 + \gamma)$$

Translation, rotational

(b) Rearrange $L = T - U$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\gamma m \dot{x}^2 \Rightarrow \frac{1}{2}m(1+\gamma)\dot{x}^2$$

$$U = \frac{1}{2}Kx^2 \qquad L = T - U$$

$$\frac{\partial L}{\partial x} = -Kx \qquad \frac{\partial L}{\partial \dot{x}} = m(1+\gamma)\dot{x}$$

$$\Rightarrow \quad m(1+\gamma)\ddot{x} + Kx = 0$$

$$\left[-\omega^2 m(1+\gamma) + K\right]x = 0$$

$$\left[\begin{array}{l} x = Ae^{i\omega t} \\ \dot{x} = i\omega x \\ \ddot{x} = -\omega^2 x \end{array}\right.$$

$$\omega^2 = \frac{K}{m(1+\gamma)} \equiv \frac{2K}{3m}$$

$$\gamma = \frac{1}{2} \qquad 1+\gamma = \frac{3}{2}$$

#3)

(a) $X = vt = (3\times10^{8}\,\frac{m}{s})(2.2\times10^{-6}\,s)$

$X = 660\ m$

(b) $X_1 = 660\ m \qquad X_2 = 10{,}000\ m$

$\gamma = \frac{X_2}{X_1} = \frac{10{,}000}{660} = 15.15$

(c) $\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta^2 = 1 - \frac{1}{\gamma^2} = 0.9956$

$\beta = 0.9978$

(d) $\beta_{12} = \frac{\beta_1 + \beta_2}{1+\beta_1\beta_2} = 0.94 < 1$

Center of Mass

(e) Before

$P_1 = (E, 0 0 + P)$

$P_2 = (E, 0 0 - P)$

$P_{12} = (2E, 0 0 0)$

$S = P_{12}^2 = 4E^2$ initial invariant mass^2

After: work in CMS

$P_3 = (m\ 0 0 0)$

$P_4 = (m\ 0 0 0)$

$P_5 = (M, 0 0 0)$

$m = 1$

$M = 90$

$P_{345} = (2m + M, 0 0 0)$

$s = P_{345}^2 = (2m + M)^2 = (2 + 90)^2$

$S = s$

$4E^2 = (2m + M)^2$

$2E = 2m + M$

$E = \frac{2m + M}{2} = \frac{2 + 90}{2} = 46\ \text{GeV}$

F $\boxed{\text{Lab System}}$

Before

$P_1 = (E, 0 0 P) \Rightarrow E^2 - P^2 = m^2$

$P_2 = (M, 0 0 0)$

$P_{12} = (E+M, 0 0 P)$

$$S = P_{12}^2 = (E+M)^2 - P^2$$

$$= E^2 + 2EM + M^2 - P^2 \qquad = m^2$$

$$S = 2EM + 2m^2$$

$S_{After} = (2m+M)^2$ same as before

$S = S \Rightarrow 2EM + 2m^2 = (2m + M)^2$

$$E = \frac{(M+2m)^2 - 2m^2}{2m} = 4231 \text{ GeV}$$

Example $pp \to ppp\bar{p}$

Before: CMS

$P_1 = (E\ 0\ 0\ p)$

$P_2 = (E\ 0\ 0\ -p)$

$P_{12} = (2E, 0 0 0)$ $\quad P_{12}^2 = s = 4E^2$

After: $P_{3456} = (4m, 0 0 0)$ $\quad s = 16m^2$

$s = s \Rightarrow 4E^2 = 16m^2 \Rightarrow E = 2m \sim 2\,\text{GeV}$

Lab frame

Before:

$P_1 = (E\ 0\ 0\ p)$

$P_2 = (m\ 0\ 0\ 0)$

$P_{12} = (E+m, 0\ 0\ p)$

$s = 2Em + 2m^2$

$s = s \Rightarrow 2Em + 2m^2 = 16m^2$

$2Em = 14m^2$

$E = 7m \hat{=} 7\,\text{GeV}$