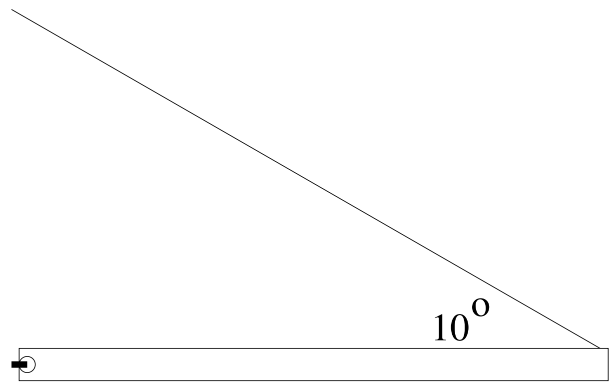
Problem 1 [20 Points]

A uniform beam of mass m and length ℓ is hinged at the left end by a frictionless pin and is held by a massless wire at the right end. If the wire is cut at time zero, how long does the rod take to fall from horizontal to vertical? Set up, but do not evaluate the definite integral. The integral should be written so clearly that a freshman could type it into Mathematica and get a numerical answer.



Problem 2 [20 Points]

A particle of mass M at rest decays into a particle of mass m and a massless particle. Find

(a) (10 points) the total energy of the massive decay product

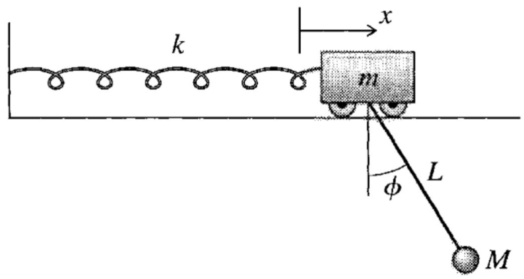
(b) (10 points) the momentum of the massless decay product

Problem 3 [30 Points]

Trying to set a speed record, a motorcyclist takes off from rest on a flat circular horizontal track. What fraction of the track will be covered before the maximum possible speed is achieved, if the motorcycle undergoes maximum acceleration without slipping while maintaining a fixed radius?

Problem 4 [30 Points]

A simple pendulum (mass M and lengh L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in the figure below.



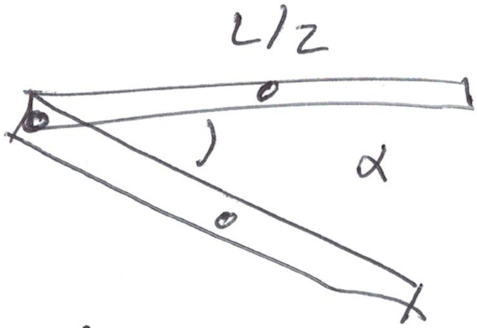
(a) Assuming that the angle ϕ remains small, write down the system's Lagrangian (10 points) and the equations of motion for x and ϕ (5 points).

(b) Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units) find the normal frequencies (5 points), the normal mode eigenvectors describing the velocities of the two masses undergoing normal oscillations (5 points), and describe the motion of the corresponding normal modes (5 points).

CPE Mechanics Jan 2021

$\boxed{\#1}$

Use Energy Conservation



$U_1 = 0 \quad K_1 = 0$

$U_2 = -mgh =$

$= -mg\frac{L}{2}\sin\theta$

$K_2 = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(\frac{1}{3} m L^2\right)\omega^2$

$\Delta E = 0 \Rightarrow \frac{1}{6} m L^2 \omega^2 = mg\frac{L}{2}\sin\theta$

$$\omega = \frac{d\theta}{dt} = \sqrt{\frac{3g\sin\theta}{L}}$$

$$\int_0^t dt = \int_{\theta=0}^{\theta=\pi/2} \sqrt{\frac{L}{3g}} \frac{d\theta}{\sqrt{\sin\theta}}$$

$$t = \sqrt{\frac{L}{3g}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \cong 2.62$$

Not required

#2

2

Before:

$$P_0 = (M\ 0\ 0\ 0)$$

$$s = P_0^2 = M^2$$

After

$$P_1 = (E\ 0\ 0\ p)$$

$$P_2 = (p\ 0\ 0\ -p)$$

$$P_{12} = (E+p,\ 0\ 0\ 0)$$

$$s = P_{12}^2 = (E+p)^2$$

$$E^2 - p^2 = m^2$$

$s = s \Rightarrow$

$$M^2 = (E+p)^2$$

$$M = (E+p)$$

$$p = M - E \qquad E^2 - p^2 = m^2$$

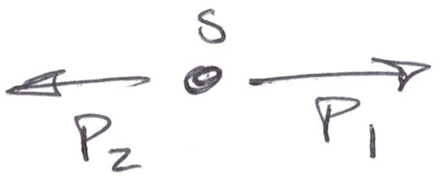
$$E^2 - (M-E)^2 = m^2$$

$$\cancel{E^2} - M^2 + 2ME + \cancel{E^2} = m^2$$

$$E = \left(\frac{M^2 + m^2}{2M}\right) c^2$$

$$\Rightarrow p = M - E = \left(\frac{M^2 - m^2}{2M}\right) c$$

Note General Process



$$P_1 = (E_1 \; 0 \; 0 \; +p) \qquad E_1^2 - p^2 = m_1^2$$

$$P_2 = (E_2 \; 0 \; 0 \; -p) \qquad E_2^2 - p^2 = m_2^2$$

$$E_{\substack{1\\2}} = \frac{s \pm m_1^2 \mp m_2^2}{2\sqrt{s}}$$

$$p = \frac{\Delta(s, m_1^2, m_2^2)}{2\sqrt{s}}$$

$$\Delta(a,b,c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)}$$

Check: $m_2 = 0 \qquad \Delta \rightarrow s - m_1^2 \longrightarrow M^2 - m_1^2$

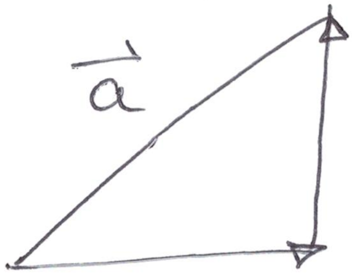
Let $s = M^2$

$$E_1 = \frac{M^2 + m_1^2}{2M} \qquad p = \frac{M^2 - m_1^2}{2M}$$

$$E_2 = \frac{M^2 - m_1^2}{2M} \equiv p$$

Check!

Motorcycle



$$a_r = \frac{v^2}{R}$$

$$a_T = \frac{dv}{dt}$$

Max Velocity:

$$F = ma$$

$$\mu N = \frac{mv^2}{r}$$

$$\mu \cancel{m} g = \frac{\cancel{m} v^2}{r}$$

$$v_{max}^2 = \mu g r$$

$$F = ma$$

$$\mu N = \mu mg = ma$$

$$\boxed{\mu g = a \quad \text{max acceleration}}$$

5

$$a = \sqrt{a_t^2 + a_r^2} = a_{max} = \mu g$$

$$= \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{R}\right)^2} = \mu g$$

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

$$\Rightarrow \frac{dv}{dt} = \sqrt{(\mu g)^2 - \left(\frac{v^2}{R}\right)^2}$$

$$\frac{dv}{ds}v = \sqrt{}$$

$$\int_{v=0}^{v=v_{max}} \frac{dv\, v}{\sqrt{}} = \int_0^S ds = S$$

$$v_{max} = \sqrt{\mu g R}$$

$$\int_0^{v_{max}} \frac{v\,dv}{\sqrt{(\mu g)^2 - \left(\frac{v^2}{R}\right)^2}}$$

Let $x = \frac{v}{v_{max}}$

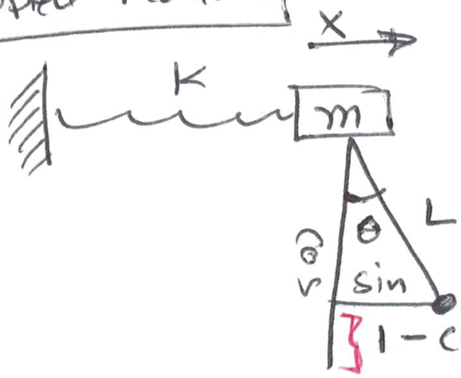
$$dx = \frac{dv}{dv_{max}}$$

$$\frac{S}{R} = \int_0^1 \frac{dx}{\sqrt{\frac{1}{x^2} - x^2}} = \pi/4$$

$$\frac{\pi/4}{2\pi} = 1/8$$

1/8'th OF track

Coupled Motion



$\sin\theta \sim \theta$

$1-\cos\theta \doteq \dfrac{\theta^2}{2}$ ← ∝ height of M

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M (\dot{x} + L\dot{\theta})^2$$

↑ horizontal velocity of M

For small angles, we ignore vertical velocity $\sim \dot{\theta}^4$

$$V = \frac{1}{2} K X^2 + \underbrace{M g L \frac{\theta^2}{2}}_{\text{height of M}}$$

$$\mathcal{L} = T - V$$

$\boxed{10 \text{ pts}}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \qquad (m+M)\ddot{x} + LM\ddot{\theta} + Kx = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad LM\ddot{x} + L^2 M\ddot{\theta} + MgL\theta = 0$$

$$T_{matrix} = \begin{pmatrix} m+M & LM \\ LM & L^2 M \end{pmatrix} \quad V_{matrix} = \begin{pmatrix} K & 0 \\ 0 & gLM \end{pmatrix}$$

$\boxed{5 \text{ pts}}$

Eq of Motion $\Rightarrow$ $V_{mat} - T_{mat}\,\omega^2 = 0$

$$\begin{pmatrix} K-(m+M)\omega^2 & -LM\omega^2 \\ -LM\omega^2 & gLM - L^2M\omega^2 \end{pmatrix} = 0$$

Set $m = M = L = g = 1$ and $K = 2$

$$\Rightarrow \begin{pmatrix} 2-2\omega^2 & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{pmatrix} = 0$$

$$\text{Det} = \omega^4 - 4\omega^2 + 2 = 0$$

$$\boxed{\omega^2 = 2 \pm \sqrt{2}}$$ Eigen frequencies

5pts

Find Normal Modes

$\omega^2 = 2 + \sqrt{2}$

$$V_{mat} - T_{mat}\,\omega^2 \longrightarrow \begin{pmatrix} \cdots & \cdots \\ -2-\sqrt{2} & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow (-2-\sqrt{2})\,a + (-1-\sqrt{2})\,b = 0$$

$$\Rightarrow b = -a\,\frac{(-2-\sqrt{2})}{(-1-\sqrt{2})}\,\frac{(-1+\sqrt{2})}{(-1+\sqrt{2})} = -\sqrt{2}\,a$$

$\therefore$ Eigenvector #1 $= v_1 = (1, -\sqrt{2})$

$\omega_1 = 2 + \sqrt{2} \cong 3.4$

$\omega_2 = 2 - \sqrt{2}$

$$V - T\omega^2 \longrightarrow \begin{pmatrix} \circ\circ\circ & \circ\circ\circ \\ -2+\sqrt{2} & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow b = -a\,\frac{(-2+\sqrt{2})}{(-1+\sqrt{2})}\,\frac{(-1-\sqrt{2})}{(-1-\sqrt{2})} = +\sqrt{2}\,a$$

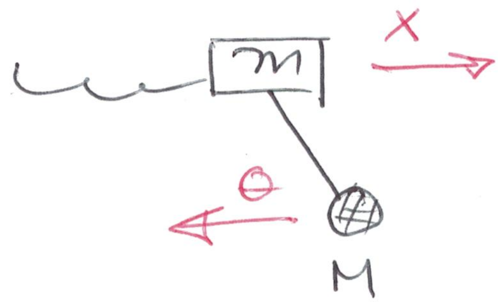
Eigenvector #2 $= v_2 = (1, +\sqrt{2})$

$\omega_2 = 2 - \sqrt{2} \cong 0.6$

$\boxed{\text{5pts}}$

#4

Mode #1: $\omega_1 \sim 3.4 > \omega_2 \sim 0.6$

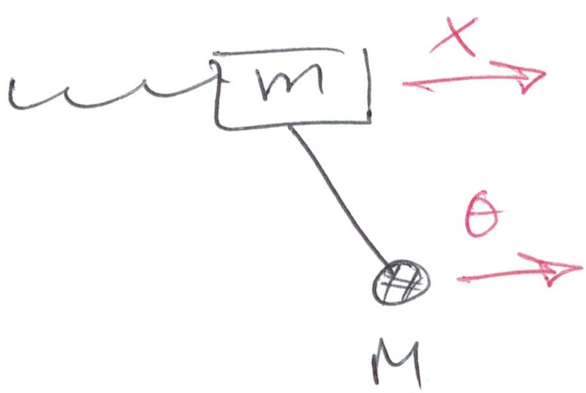


$v_1 = (1, -\sqrt{2})$

Anti-symmetric

- Motion is in opposite directions
- Frequency is LARGE

Mode #2 $\omega_2 \sim 0.6 < \omega_1 \sim 3.4$



$v_2 = (1, +\sqrt{2})$

Symmetric

- Motion is in same direction
- Frequency is small

$\boxed{\text{5pts}}$