Classical Mechanics Core Proficiency Exam
October 2021
Problems provided by the Physics Faculty.
Exam assembled by the CPE Committee

Instructions

The exam consists of two longer questions and two shorter questions.
All four problems will be graded. You have two hours to work on the solutions.
You are allowed one textbook of your choice, one math reference, and a calculator
(no cell phone calculator allowed).

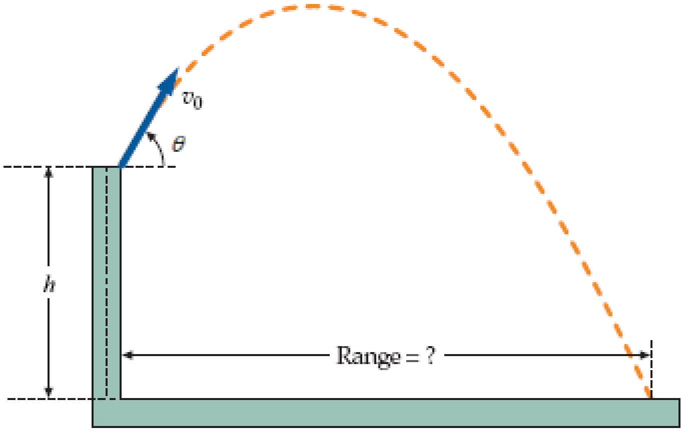
Write your personal identifier (e.g. name, number, etc.) on the cover sheet of this exam.

We will scan the exams to PDF, so please write only on one side of the page.
Sort your copy of the exam together with your scratch paper (exam on top) with the problems in numerical order. You may wish to number the pages so we can check the scanner did not miss a page.

Problem	Points	Score			Final Score
1	20				
2	20				
3	30				
4	30				
Total	100				

Problem #1 (20 points)

In the middle of a level piece of ground, a tower of height h is constructed. A cannon with muzzle speed v_0 is carried to the top of the tower. Assume that the acceleration due to gravity is a constant g, and neglect air resistance. Express all answers in terms of the given quantities. Find the maximum range R of the cannon ball.



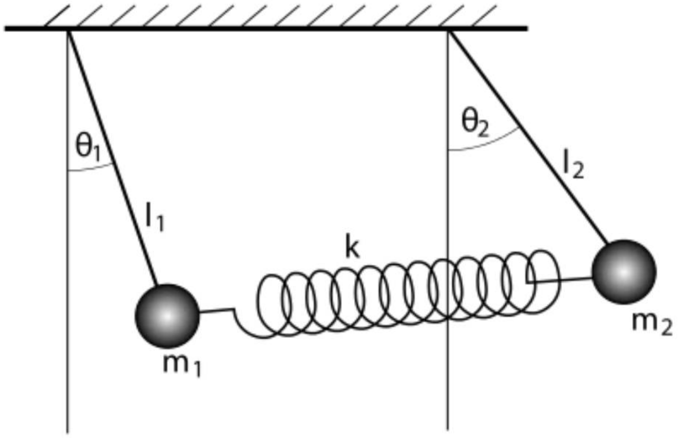
Problem #2 (20 points)

In galaxies like our own, the visible stars and gas appear to be embedded in a much more massive cloud of invisible "dark matter" that interacts little if at all with ordinary matter and electromagnetic radiation. If the dark matter consists of X-particles of mass M, and if these particles possess non-gravitational interactions of some kind, there are several ways that we might detect them.

We may try to produce dark matter particles in an electron-positron collider, in which electrons and positrons of **unequal** lab frame energies E_1 and E_2 collide head-on. If an electron-positron collision produces a pair of X's via $e^+ + e^- \rightarrow X+X$, and assuming that the energies involved are large enough that we can neglect the masses of the electrons and positrons, what is the largest mass M of the X particle that this collider can produce?

Problem #3 (30 points)

Two pendula, each of which consists of a weightless rigid rod length of $L_1=L_2=L$ and masses $m_1=m_2=m$ are connected at their endpoints by a spring with spring constant k. Consider only small displacements from equilibrium.



a) (10 points) Write down the Lagrangian, and the Lagrange equations of motion.

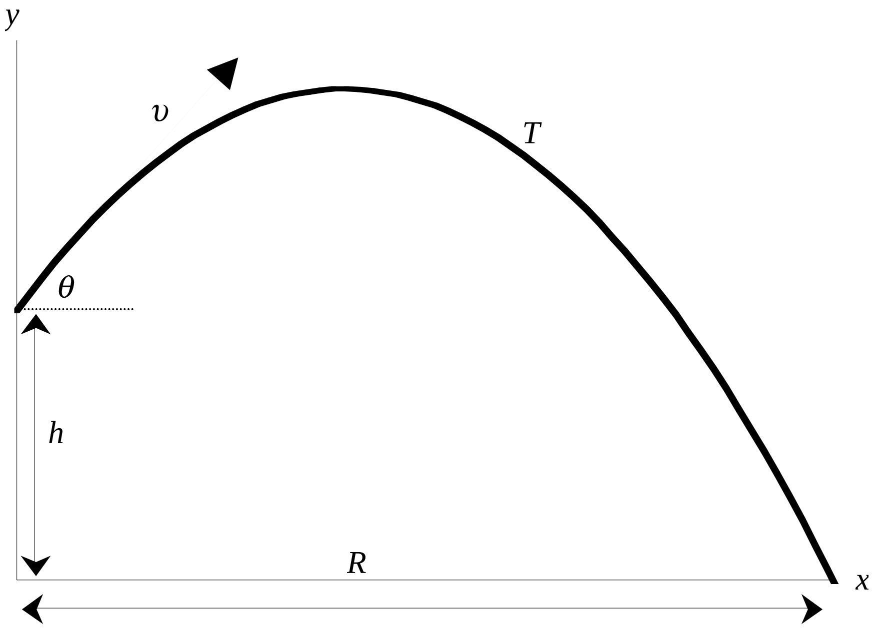
b) (10 points) What are the normal modes of this system AND frequencies of the normal modes? Briefly describe these modes.

c) (10 points) At t=0 the right pendulum is displaced by an angle θ_2 to the right while the left pendulum remains vertical (θ_1=0). Both pendula are released from rest. Describe the subsequent motion.

Maximum Range of a Projectile Launched from a Height—C.E. Mungan, Spring 2003

reference: TPT 41:132 (March 2003)

Find the launch angle θ and maximum range R of a projectile launched from height h at speed υ.



The basic equations of kinematics at the landing point after flight time T are

$$0 = h + \upsilon_y T - \tfrac{1}{2} g T^2 \tag{1}$$

vertically and

$$R = \upsilon_x T \tag{2}$$

horizontally. Substitute Eq. (2) for T into (1) and convert from rectangular to polar components to get

$$h(1 + \cos 2\theta) = \frac{gR^2}{\upsilon^2} - R\sin 2\theta. \tag{3}$$

Maximize R by differentiating this expression with respect to θ and putting $dR/d\theta = 0$ to obtain an expression for the optimum launch angle,

$$\tan 2\theta = \frac{R}{h} \quad \Rightarrow \quad \theta = \frac{1}{2}\tan^{-1}\frac{R}{h}. \tag{4}$$

This implies $\cos 2\theta = h(h^2 + R^2)^{-1/2}$ and $\sin 2\theta = R(h^2 + R^2)^{-1/2}$. Substitute these into Eq. (3) to obtain the maximum range,

$$h = \frac{gR^2}{2\upsilon^2} - \frac{\upsilon^2}{2g} \quad \Rightarrow \quad R = \sqrt{\frac{2\upsilon^2}{g}\left(h + \frac{\upsilon^2}{2g}\right)}. \tag{5}$$

Equations (4) and (5) can be normalized for plotting purposes in terms of

$$R_0 \equiv \frac{v^2}{g}, \tag{6}$$

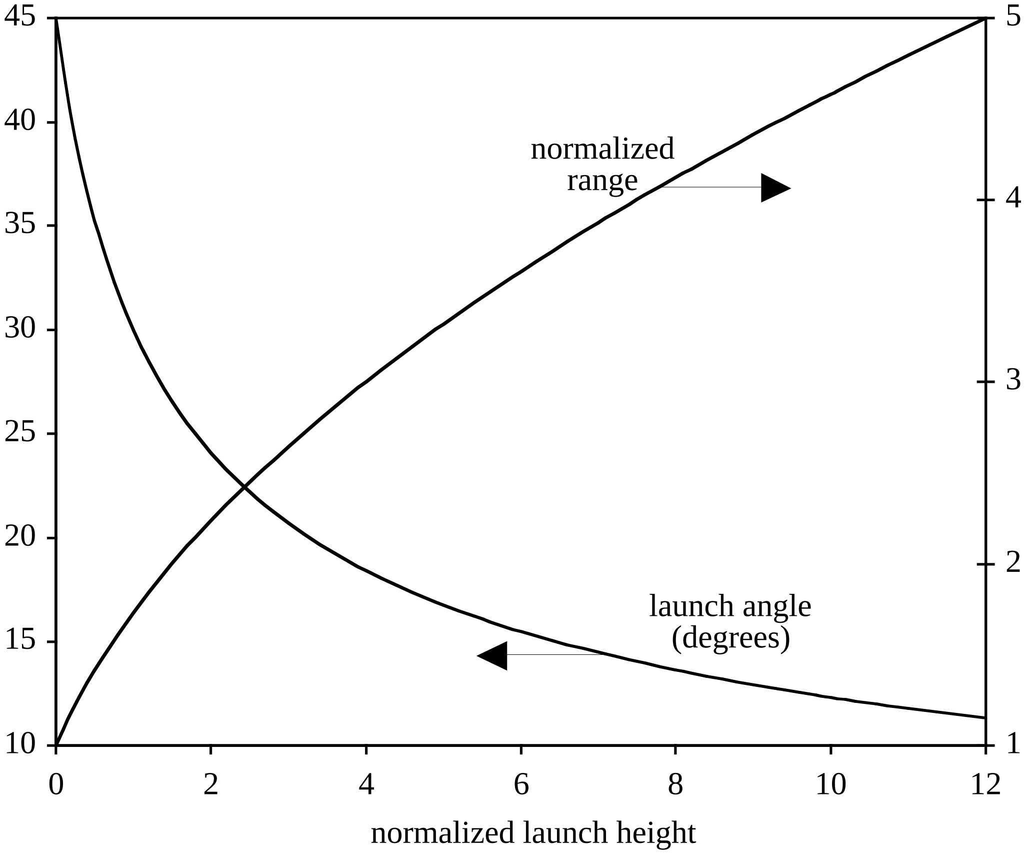
the maximum range for a surface-to-surface projectile (i.e., when $h = 0$), to get the normalized range

$$\frac{R}{R_0} = \sqrt{1 + 2\frac{h}{R_0}} \tag{7}$$

at a launch angle of

$$\theta = \frac{1}{2}\sec^{-1}\left(1 + \frac{R_0}{h}\right). \tag{8}$$

These two equations are plotted below as a function of the normalized launch height h / R_0.



As expected, $R = R_0$ and $\theta = 45^\circ$ when $h = 0$. In the other limit, $R \propto h^{1/2}$ and $\theta \to 0^\circ$ as $h \to \infty$. More reasonably, notice that if you launch from a height equal to 1.5 times your surface range, you can get the projectile to go twice as far, provided you launch it at 26.6° (half of a 3–4–5 triangle angle).

Dark Matter Problem :

Ⓐ $e^+ + e^- \to X + X$ neglect $e^{\pm}$ mass

Mass $X = M$

Relativistically :

$$P_1^\mu + P_2^\mu \to P_3^\mu + P_4^\mu$$

$$\left.\begin{array}{l} P_1^\mu = (E_1 \ 0 \ 0 \ E_1) \\ P_2^\mu = (E_2 \ 0 0 \ \underset{\uparrow}{-}E_2) \end{array}\right) \text{Zero mass assummed}$$

colliding

$$P_{12}^\mu = (E_1 + E_2 \ 0 0 \ E_1 - E_2)$$

$$P_{12}^2 = S = (E_1 + E_2)^2 - (E_1 - E_2)^2 = 4 E_1 E_2$$

Final State : work in CMS w/ invariants

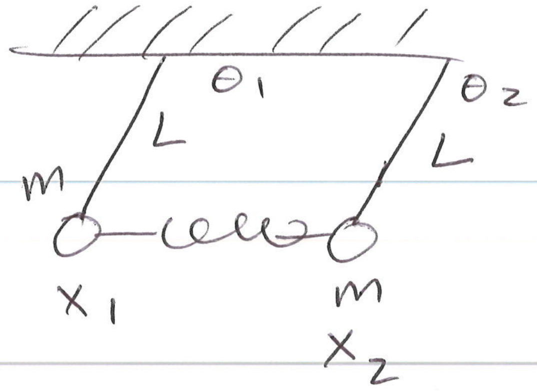
$$P_3^\mu = (M \ 0 0 0)$$

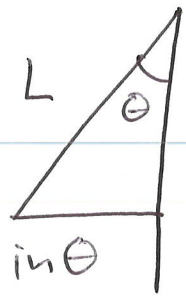
$$P_4^\mu = (M \ 0 0 0)$$

$$P_{34}^\mu = (2M \ 0 0 0)$$

$$P_{34}^2 = S = 4M^2 \equiv 4 E_1 E_2$$

$$M = \sqrt{E_1 E_2}$$





$L(1-\cos\theta) \simeq L\frac{\theta^2}{2}$

$L\sin\theta \sim L\theta$

$$T = \frac{1}{2} m v^2 = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{mL^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$V = mgh \longrightarrow mg\left[L\frac{\theta_1^2}{2} + L\frac{\theta_2^2}{2}\right]$$

$$+ \frac{1}{2} K \Delta x^2 \longrightarrow \frac{K}{2}\left(L\theta_1 - L\theta_2\right)^2$$

$$V \rightarrow \frac{mgL}{2}\left[\theta_1^2 + \theta_2^2\right] + \frac{KL^2}{2}\underbrace{\left(\theta_1 - \theta_2\right)^2}_{\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2}$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = mL^2 \ddot{\theta}_1 \qquad -\frac{\partial L}{\partial \theta_1} = KL^2(\theta_1 - \theta_2) + mgL\theta_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = mL^2 \ddot{\theta}_2 \qquad -\frac{\partial L}{\partial \theta_2} = KL^2(\theta_2 - \theta_1) + mgL\theta_2$$

$$\boxed{\underline{M}\ddot{X} + \underline{K}X = 0}$$

$$mL^2\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_1\end{pmatrix} + KL^2\begin{pmatrix}1&-1\\-1&1\end{pmatrix} + mgL\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

divide by mL^2

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} + \begin{pmatrix}\frac{K}{m}+\frac{g}{L} & -\frac{K}{m}\\ -\frac{K}{m} & \frac{K}{m}+\frac{g}{L}\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = 0$$

$$\boxed{(\underline{K} - \omega^2\underline{M})\vec{v} = 0}$$

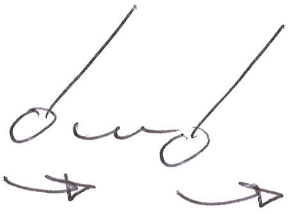
$$\begin{pmatrix}\frac{K}{m}+\frac{g}{L}-\omega^2 & -\frac{K}{m}\\ -\frac{K}{m} & \frac{K}{m}+\frac{g}{L}-\omega^2\end{pmatrix}$$

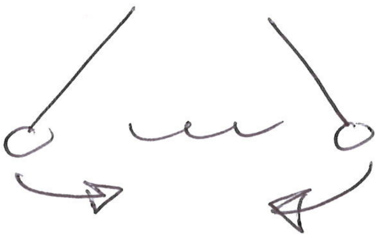
Solve Det( ) = 0

$$\Rightarrow \omega^2 = \frac{g}{L} \quad \text{or} \quad \frac{2K}{m}+\frac{g}{L}$$

Simple pendulum
No spring motion
Symmetric Mode

Anti-Symm.
w/ Spring
Oscillation





By Symmetry:

Modes: $v_1 = (1\ 1)$ $\qquad \omega^2 = g/L$

$v_2 = (1\ -1)$ $\qquad \omega^2 = \frac{2k}{m} + g/L$

$\boxed{\text{Check}}$ $\omega_1^2 \to g/L$

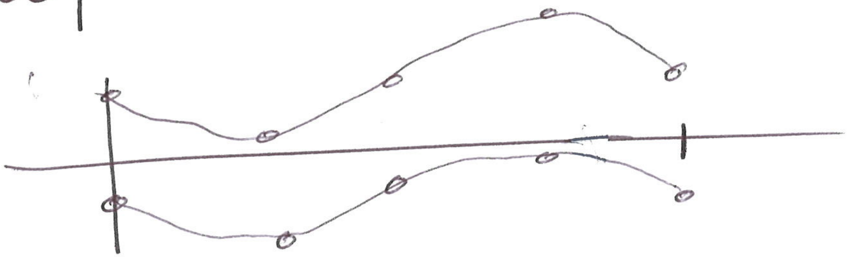
$() \to \frac{k}{m}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \circ v_1 = \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = 0$

$\boxed{\text{Check}}$ $\omega_2^2 \to \frac{2k}{m} + g/L$

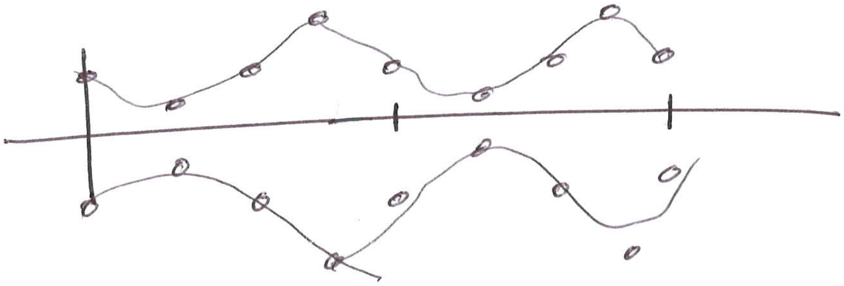
$() \to \frac{k}{m}\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix} \circ v_2 = \begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = 0$

$\omega_2 > \omega_1$

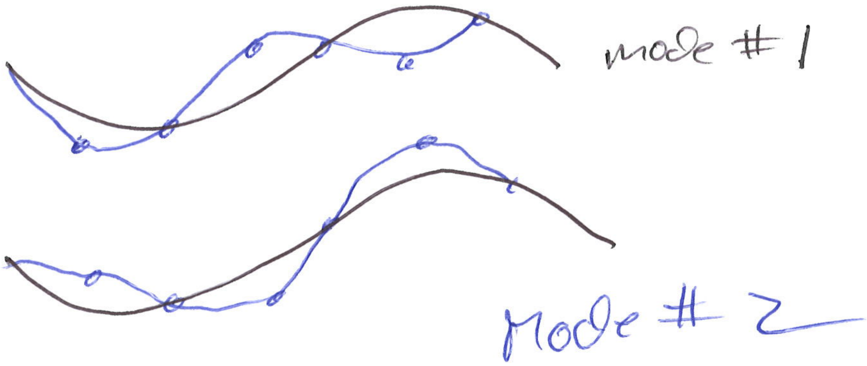
Mode #1

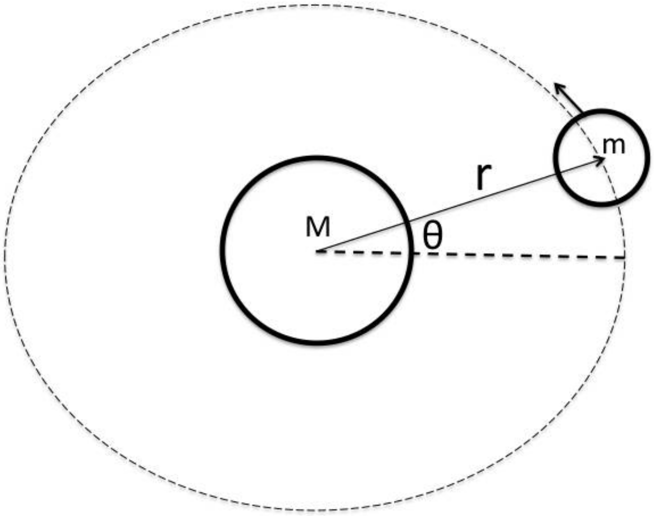


Mode #2



Combined





Problem #4 (30 points)

An object of mass m is orbiting around another object of mass M under a central potential V=V(r), where r is the distance between two objects. Here, we assume m<<M and the angular position of m is θ.

a) (8 points) Write the Lagrangian of the mass m in terms of r and θ, and obtain the equations of the motion. Show that the angular momentum L and the areal velocity (the area swept out by the radius vector per unit time) are conserved.

b) (8 points) Given the angular momentum L, derive the expression for the effective potential $V_{eff}(r)$ subject to mass m as a function of r.

c) (8 points) Let us assume that $V(r) = k\,r^n$. Find the condition of n and k such that the orbit can be circular. Derive the radius R (R > 0) of this circular orbit in terms of k, n, and L.

d) (6 points) Find the conditions on n and k such that the circular orbit is stable under perturbation in the radial direction.

3. a) As the reduced mass $\mu = \frac{M \cdot m}{M+m} \simeq m$, the Lagrangian L is given by

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{20}$$

The equations of the motion satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{21}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{22}$$

The angular momentum $l = mr^2\dot{\theta}$ satisfies $\dot{l} = 0$, therefore it is conserved. The areal velocity $\dot{A} = \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$, therefore it is also conserved.

3. b) Using $l = mr^2\dot{\theta}$,

$$m\ddot{r} = -\left(\frac{\partial V}{\partial r} - mr\dot{\theta}^2\right) = -\left(\frac{\partial V}{\partial r} - \frac{l^2}{mr^3}\right) \tag{23}$$

Since $m\ddot{r} = -\frac{\partial V_{eff}}{\partial r}$, the effective potential V_{eff} is given by

$$V_{eff}(r) = V(r) + \frac{l^2}{2mr^2}. \tag{24}$$

3. c) When $V(r) = kr^n$, the condition for the circular orbit can be satisfied if $\ddot{r} = 0$ at the orbit radius R ($R > 0$):

$$\frac{d}{dr}V_{eff}(r)\Big|_{r=R} = nkR^{n-1} - \frac{l^2}{mR^3} = 0 \tag{25}$$

The above equation can be satisfied when $nk > 0$ ($n \neq -2$) and the radius R is given by

$$R^{n+2} = \frac{l^2}{nkm}, \quad R = \left(\frac{l^2}{nkm}\right)^{\left(\frac{1}{n+2}\right)}. \tag{26}$$

3. d) The condition for the stability is given by $\frac{d^2}{dr^2}V_{eff}(r)\Big|_{r=R} > 0$. Using the relation $nkR^{n-2} = \frac{l^2}{mR^4}$,

$$n(n-1)kR^{n-2} + 3\frac{l^2}{mR^4} = \frac{(n+2)l^2}{mR^4} > 0 \tag{27}$$

The above condition is satisfied when $n > -2$ and $nk > 0$.