ONE TWO THREE... INFINITY

Facts and Speculations of Science

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to the world. That was three centuries ago, and ever since then the best mathematicians in each country have tried to reconstruct the proof that Fermat had in mind when he wrote his marginal note. But up to the present time no proof has been discovered. To be sure, considerable progress has been made toward the ultimate goal, and an entirely new branch of mathematics, the so-called "theory of ideals," has been created in attempts to prove Fermat's theorem. Euler demonstrated the impossibility of integer solution of the equations: $x^3+y^3=z^3$ and $x^4+y^4=z^4$, Dirichlet proved the same for the equation: $x^5+y^5=z^5$, and through the combined efforts of several mathematicians we now have proofs that no solution of the Fermat equation is possible when n has any value smaller than 269. Yet no general proof, good for *any* values of the exponent n, has ever been achieved, and there is a growing suspicion that Fermat himself either did not have any proof or made a mistake in it. The problem became especially popular when a prize of a hundred thousand German marks was offered for its solution, though of course all the efforts of money-seeking amateurs did not accomplish anything.

The possibility, of course, always remains that the theorem is wrong and that an example can be found in which the sum of two equal high powers of two integers is equal to the same power of a third integer. But since in looking for such an example one must now use only exponents larger than 269, the search is not an easy one.

2. *THE MYSTERIOUS* $\sqrt{-1}$

Let us now do a little advanced arithmetic. Two times two are four, three times three are nine, four times four are sixteen, and five times five are twenty-five. Therefore: the square root of four is two, the square root of nine is three, the square root of sixteen is four, and the square root of twenty-five is five.[4]

But what would be the square root of a negative number? Have expressions like $\sqrt{-5}$ and $\sqrt{-1}$ any meaning?

[4] It is also easy to find the square roots of many other numbers. Thus, for example, $\sqrt{5}=2.236\ldots$ because: $(2.236\ldots)\times(2.236\ldots)=5.000\ldots$ and $\sqrt{7.3}=2.702\ldots$ because: $(2.702\ldots)\times(2.702\ldots)=7.300\ldots$

If you try to figure it out in a rational way, you will undoubtedly come to the conclusion that the above expressions make no sense at all. To quote the words of the twelfth century mathematician Brahmin Bhaskara: "The square of a positive number, as also that of a negative number, is positive. Hence the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for a negative number is not a square."

But mathematicians are obstinate people, and when something that seems to make no sense keeps popping up in their formulas, they will do their best to put sense into it. And the square roots of negative numbers certainly do keep popping up in all kinds of places, whether in the simple arithmetical questions that occupied mathematicians of the past, or in the twentieth century problem of unification of space and time in the frame of the theory of relativity.

The brave man who first put on paper a formula that included the apparently meaningless square root of a negative number was the sixteenth century Italian mathematician Cardan. In discussing the possibility of splitting the number 10 into two parts the product of which would be 40, he showed that, although this problem does not have any rational solution, one could get the answer in the form of two impossible mathematical expressions: $5+\sqrt{-15}$ and $5-\sqrt{-15}$.[5]

Cardan wrote the above lines with the reservation that the thing is meaningless, fictitious, and imaginary, but still he wrote them.

And if one dares to write square roots of negatives, imaginary as they may be, the problem of splitting the number 10 into the two desired parts can be solved. Once the ice was broken the square roots of negative numbers, or imaginary numbers as they were called after one of Cardan's epithets, were used by various mathematicians more and more frequently, although always with great reservations and due excuses. In the book on algebra pub-

[5] The proof follows:

$(5+\sqrt{-15})+(5-\sqrt{-15})=5+5=10$ and

$$(5+\sqrt{-15})\times(5-\sqrt{-15})=(5\times 5)+5\sqrt{-15}-5\sqrt{-5}-(\sqrt{-15}\times\sqrt{-15})$$
$$=(5\times 5)-(-15)=25+15=40.$$

lished in 1770 by the famous German mathematician Leonard Euler we find a large number of applications of imaginary numbers, mitigated however, by the comment: "All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc. are impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible."

But in spite of all these abuses and excuses imaginary numbers soon became as unavoidable in mathematics as fractions, or radicals, and one could practically not get anywhere without using them.

The family of imaginary numbers represents, so to speak, a fictitious mirror image of the ordinary or real numbers, and, exactly in the same way as one can produce all real numbers starting with the basic number 1, one can also build up all imaginary numbers from the basic imaginary unit $\sqrt{-1}$, which is usually denoted by the symbol *i*.

It is easy to see that $\sqrt{-9}=\sqrt{9}\times\sqrt{-1}=3i$; $\sqrt{-7}=\sqrt{7}\cdot\sqrt{-1}=2.646\ldots i$ etc., so that each ordinary real number has its imaginary double. One can also combine real and imaginary numbers to make single expressions such as $5+\sqrt{-15}=5+i\sqrt{15}$ as it was first done by Cardan. Such hybrid forms are usually known as complex numbers.

For well over two centuries after imaginary numbers broke into the domain of mathematics they remained enveloped by a veil of mystery and incredibility until finally they were given a simple geometrical interpretation by two amateur mathematicians: a Norwegian surveyor by the name of Wessel and a Parisian bookkeeper, Robert Argand.

According to their interpretation a complex number, as for example $3+4i$, may be represented as in Figure 10, in which 3 corresponds to the horizontal distance, and 4 to the vertical, or ordinate.

Indeed all ordinary real numbers (positive or negative) may be represented as corresponding to the points on the horizontal axis, whereas all purely imaginary ones are represented by the points on the vertical axis. When we multiply a real number, say 3, representing a point on the horizontal axis, by the imaginary unit i we obtain the purely imaginary number $3i$, which must be plotted on the vertical axis. Hence, *the multiplication by i is geometrically equivalent to a counterclockwise rotation by a right angle.* (See Figure 10).

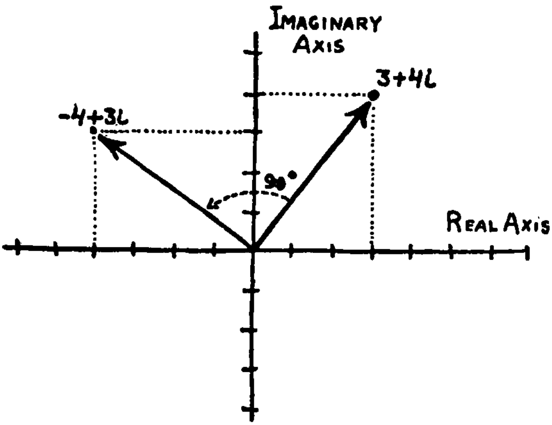

FIGURE 10

If now we multiply $3i$ once more by i, we must turn the thing by another 90 degrees, so that the resulting point is again brought back to the horizontal axis, but is now located on the negative side. Hence,

$$3i \times i = 3i^2 = -3, \text{ or } i^2 = -1.$$

Thus the statement that the "square of i is equal to -1" is a much more understandable statement than "turning twice by a right angle (both turns counterclockwise) you will face in the opposite direction."

The same rule also applies, of course, to hybrid complex numbers. Multiplying $3+4i$ by i we get:

$$(3+4i)\, i = 3i + 4i^2 = 3i - 4 = -4 + 3i.$$

And as you can see at once from Figure 10, the point $-4+3i$ corresponds to the point $3+4i$, which is turned counterclockwise by 90 degrees around the origin. Similarly the multiplication by

$-i$ is nothing but the clockwise rotation around the origin, as can be seen from Figure 10.

If you still feel a veil of mystery surrounding imaginary numbers you will probably be able to disperse it by working out a simple problem in which they have practical application.

There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:

"Sail to ______ North latitude and ______ West longitude[6] where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine.[7] There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn *right* by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn *left* by a right angle and see that thou takest the same number of steps, and put another spike into the ground. Dig halfway between the spikes; the treasure is there."

The instructions were quite clear and explicit, so our young man chartered a ship and sailed to the South Seas. He found the island, the field, the oak and the pine, but to his great sorrow the gallows was gone. Too long a time had passed since the document had been written; rain and sun and wind had disintegrated the wood and returned it to the soil, leaving no trace even of the place where it once had stood.

Our adventurous young man fell into despair, then in an angry frenzy began to dig at random all over the field. But all his efforts were in vain; the island was too big! So he sailed back with empty hands. And the treasure is probably still there.

A sad story, but what is sadder still is the fact that the fellow might have had the treasure, if only he had known a bit about

[6] The actual figures of longitude and latitude were given in the document but are omitted in this text, in order not to give away the secret.

[7] The names of the trees are also changed for the same reason as above. Obviously there would be other varieties of trees on a tropical treasure island.

mathematics, and specifically the use of imaginary numbers. Let us see if we can find the treasure for him, even though it is too late to do him any good.

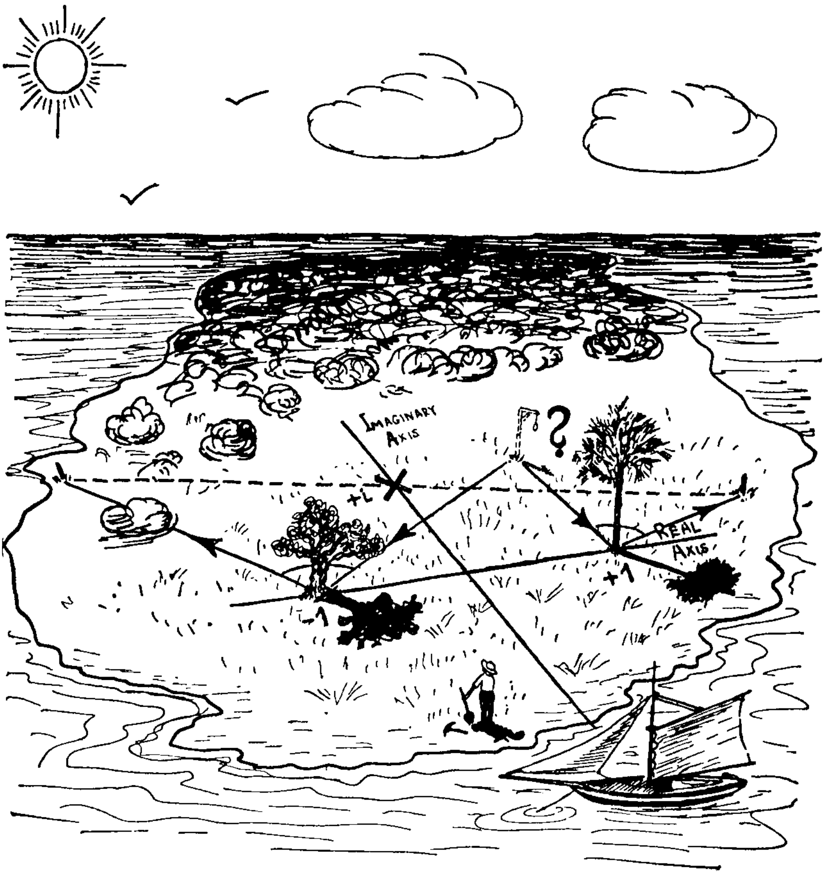

FIGURE 11

Treasure hunt with imaginary numbers.

Consider the island as a plane of complex numbers; draw one axis (the real one) through the base of the two trees, and another axis (the imaginary one) at right angles to the first, through a point half way between the trees (Figure 11). Taking one half of the distance between the trees as our unit of length, we can say that the oak is located at the point -1 on the real axis, and the pine at the point $+1$. We do not know where the gallows was so let us denote its hypothetical location by the Greek letter Γ (capital gamma), which even looks like a gallows. Since the gallows was not necessarily on one of the two axes Γ must be considered as a complex number: $\Gamma = a + bi$, in which the meaning of a and b is explained by Figure 11.

Now let us do some simple calculations remembering the rules of imaginary multiplication as stated above. If the gallows is at Γ and the oak at -1, their separation in distance and direction may be denoted by $(-1) - \Gamma = -(1+\Gamma)$. Similarly the separation of the gallows and the pine is $1 - \Gamma$. To turn these two distances by right angles clockwise (to the right) and counter-clockwise (to the left) we must, according to the above rules multiply them by $-i$ and by i, thus finding the location at which we must place our two spikes as follows:

$$\text{first spike: } (-i)[-(1+\Gamma)] + 1 = i(\Gamma+1) - 1$$
$$\text{second spike: } (+i)(1-\Gamma) - 1 = i(1-\Gamma) + 1$$

Since the treasure is halfway between the spikes, we must now find one half the sum of the two above complex numbers. We get:

$$\tfrac{1}{2}[i(\Gamma+1) + 1 + i(1-\Gamma) - 1] = \tfrac{1}{2}[+i\Gamma + i + 1 + i - i\Gamma - 1]$$
$$= \tfrac{1}{2}(+2i) = +i.$$

We now see that the unknown position of the gallows denoted by Γ fell out of our calculations somewhere along the way, and that, regardless of where the gallows stood, the treasure must be located at the point $+i$.

And so, if our adventurous young man could have done this simple bit of mathematics, he would not have needed to dig up the entire island, but would have looked for the treasure at the point indicated by the cross in Figure 11, and there would have found the treasure.

If you still do not believe that it is absolutely unnecessary to know the position of the gallows in order to find the treasure, mark on a sheet of paper the positions of two trees, and try to carry out the instructions given in the message on the parchment by assuming several different positions for the gallows. You will always get the same point, corresponding to the number $+i$ on the complex plane!

Another hidden treasure that was found by using the imaginary square root of -1 was the astonishing discovery that our ordinary three-dimensional space and time can be united into one four-dimensional picture governed by the rules of four-dimensional geometry. But we shall come back to this discovery in one of the following chapters, in which we discuss the ideas of Albert Einstein and his theory of relativity.