Eigen System

$$M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \qquad \vec{M}\vec{v} = \lambda \vec{v}$$

$$M - \lambda \cong \begin{vmatrix} (3-\lambda) & -1 \\ -1 & (3-\lambda) \end{vmatrix} = (3-\lambda)^2 - 1 = 8 - 6\lambda + \lambda^2$$

$$= (\lambda - 2)(\lambda - 4)$$

$$\lambda = \{2, 4\}$$

Note: $\mathrm{Tr}\, M = 6 = 2 + 4$

Find Eigenvectors

$$M v = \lambda v \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x - y \\ -x + 3y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} 3x - y = \lambda x \\ -x + 3y = \lambda y \end{array}$$

For $\lambda = 2 \Rightarrow 3x - y = 2x \Rightarrow x = y$

$v_2 = (1, 1)$ or normalized $\frac{1}{\sqrt{2}}(1, 1)$

For $\lambda = 4 \Rightarrow 3x - y = 4x \Rightarrow x = -y$

$v_4 = (1, -1)$ or $\frac{1}{\sqrt{2}}(1, -1)$

B is matrix of eigenvectors

$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Then $B^T M B$ is diag

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T \underbrace{\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\begin{pmatrix} 2 & 4 \\ 2 & -4 \end{pmatrix}} \frac{1}{\sqrt{2}}$$

$$\frac{1}{2} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_4 \end{pmatrix} = D$$

Note $B B^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

also $\Rightarrow B^T B$

orthogonal $A^{-1} = A^T \Rightarrow A^T A = \mathbb{1}$

$$M v = \lambda v$$

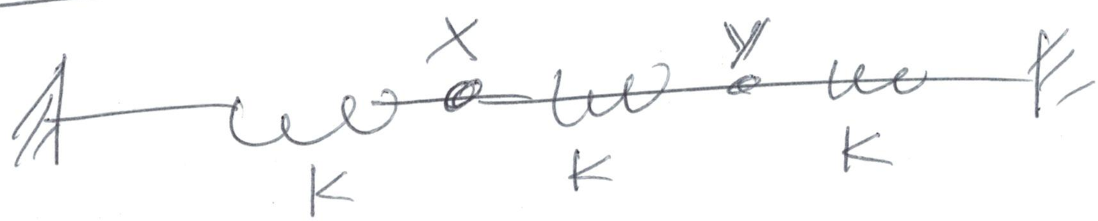
$$B^T \left[M B B^T v = \lambda v \right]$$

$$(B^T M B)(B^T v) = \lambda (B^T v)$$

$$M' \quad v' = \lambda v'$$

$$\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_4 \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \lambda \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$$

Example #3 P. 165 Boas



$$V = \frac{1}{2}Kx^2 + \frac{1}{2}K(x-y)^2 + \frac{1}{2}Ky^2$$
$$= K\left[x^2 + y^2 - xy\right]$$

Potential Energy

$$V = \frac{1}{2}Kx^2 \qquad \frac{\partial V}{\partial x} = Kx$$

Hooke's Law $F = -Kx = -\frac{\partial V}{\partial x}$

For oscillation

$$x(t) = x_0 e^{i\omega t}$$
$$\dot{x} = i\omega x$$
$$\ddot{x} = -\omega^2 x$$

$$m\ddot{x} = -\frac{\partial V}{\partial x} = -2Kx + Ky \equiv -m\omega^2 x$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = Kx - 2Ky \equiv -m\omega^2 y$$

Then

$$\begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = \overbrace{\left[+ \frac{m\omega^2}{k} \right]}^{\lambda} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \lambda = 1, 3 \qquad v_1 = (1, 1) \quad v_2 = (1, -1)$$

B = matrix of eigenvectors

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\begin{pmatrix} x + y \\ x - y \end{pmatrix}}{\sqrt{2}} \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

New coordinates

Choose $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} X+Y \\ X-Y \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1 + z_2 \\ z_1 - z_2 \end{pmatrix}$$

$$V = K\left[X^2 + Y^2 - XY\right]$$

$$= \frac{K}{2}\left[(z_1+z_2)^2 + (z_1-z_2)^2 - (z_1+z_2)(z_1-z_2)\right]$$

$$= \frac{1}{2}K\left[z_1^2 + 3z_2^2\right]$$

$$= \frac{1}{2}(1K)z_1^2 + \frac{1}{2}(3K)z_2^2$$

λ_1 λ_3