▶ **Example 3.** Find the characteristic vibration frequencies for the system of masses and springs shown in Figure 12.1.

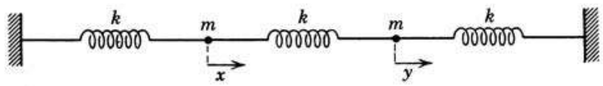


**Figure 12.1**

Let $x$ and $y$ be the coordinates of the two masses at time $t$ relative to their equilibrium positions, as shown in Figure 12.1. We want to write the equations of motion (mass times acceleration = force) for the two masses (see Chapter 2, end of Section 16). We *can* just write the forces by inspection as we did in Chapter 2, but for more complicated problems it is useful to have a systematic method. First write the potential energy; for a spring this is $V = \frac{1}{2}ky^2$ where $y$ is the compression or extension of the spring from its equilibrium length. Then the force exerted on a mass attached to the spring is $-ky = -dV/dy$. If $V$ is a function of two (or more) variables, say $x$ and $y$ as in Figure 12.1, then the forces on the two masses are $-\partial V/\partial x$ and $-\partial V/\partial y$ (and so on for more variables). For Figure 12.1, the extension or compression of the middle spring is $x - y$ so its potential energy is $\frac{1}{2}k(x-y)^2$. For the other two springs, the potential energies are $\frac{1}{2}kx^2$ and $\frac{1}{2}ky^2$ so the total potential energy is

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x-y)^2 + \frac{1}{2}ky^2 = k(x^2 - xy + y^2). \tag{12.10}$$

In writing the equations of motion it is convenient to use a dot to indicate a time derivative (as we often use a prime to mean an $x$ derivative). Thus $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, etc. Then the equations of motion are

$$\begin{cases} m\ddot{x} = -\partial V/\partial x = -2kx + ky, \\ m\ddot{y} = -\partial V/\partial y = \phantom{-2}kx - 2ky. \end{cases} \tag{12.11}$$

In a *normal* or *characteristic* mode of vibration, the $x$ and $y$ vibrations have the same frequency. As in Chapter 2, equations (16.22), we assume solutions $x = x_0e^{i\omega t}$, $y = y_0e^{i\omega t}$, with the same frequency $\omega$ for both $x$ and $y$. [Or, if you prefer, we could replace $e^{i\omega t}$ by $\sin\omega t$ or $\cos\omega t$ or $\sin(\omega t + \alpha)$, etc.] Note that (for any of these solutions),

$$\ddot{x} = -\omega^2 x, \quad \text{and} \quad \ddot{y} = -\omega^2 y. \tag{12.12}$$

Substituting (12.12) into (12.11) we get (Problem 10)

$$\begin{cases} -m\omega^2 x = -2kx + ky, \\ -m\omega^2 y = \phantom{-2}kx - 2ky. \end{cases} \tag{12.13}$$

In matrix form these equations are

$$\lambda\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \lambda = \frac{m\omega^2}{k}. \tag{12.14}$$

Note that this is an eigenvalue problem (see Section 11). To find the eigenvalues $\lambda$, we write

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \tag{12.15}$$

and solve for $\lambda$ to find $\lambda = 1$ or $\lambda = 3$. Thus [by the definition of $\lambda$ in (12.14)] the characteristic frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{3k}{m}}. \tag{12.16}$$

The eigenvectors (not normalized) corresponding to these eigenvalues are:

$$\text{For } \lambda = 1\text{: } y = x \text{ or } \mathbf{r} = (1,1); \text{ for } \lambda = 3\text{: } y = -x \text{ or } \mathbf{r} = (1,-1). \tag{12.17}$$

Thus at frequency $\omega_1$ (with $y = x$), the two masses oscillate back and forth together like this $\rightarrow\rightarrow$ and then like this $\leftarrow\leftarrow$. At frequency $\omega_2$ (with $y = -x$), they oscillate in opposite directions like this $\leftarrow\rightarrow$ and then like this $\rightarrow\leftarrow$. These two especially simple ways in which the system can vibrate, each involving just one vibration frequency, are called the characteristic (or normal) modes of vibration; the corresponding frequencies are called the characteristic (or normal) frequencies of the system.

The problem we have just done shows an important method which can be used in many different applications. There are numerous examples of vibration problems in physics—in acoustics: the vibrations of strings of musical instruments, of drumheads, of the air in organ pipes or in a room; in mechanics and its engineering applications: vibrations of mechanical systems all the way from the simple pendulum to complicated structures like bridges and airplanes; in electricity: the vibrations of radio waves, of electric currents and voltages as in a tuned radio; and so on. In such problems, it is often useful to find the characteristic vibration frequencies of the system under consideration and the characteristic modes of vibration. More complicated vibrations can then be discussed as combinations of these simpler normal modes of vibration.

▶ **Example 4.** In Example 3 and Figure 12.1, the two masses were equal and all the spring constants were the same. Changing the spring constants to different values doesn't cause any problems but when the masses are different, there is a possible difficulty which we want to discuss. Consider an array of masses and springs as in Figure 12.1 but with the following masses and spring constants: $2k$, $2m$, $6k$, $3m$, $3k$. We want to find the characteristic frequencies and modes of vibration. Following our work in Example 3, we write the potential energy $V$, find the forces, write the equations of motion, and substitute $\ddot{x} = -\omega^2 x$, and $\ddot{y} = -\omega^2 y$, in order to find the characteristic frequencies. (*Do the details*: Problem 11.)

$$V = \frac{1}{2}2kx^2 + \frac{1}{2}6k(x-y)^2 + \frac{1}{2}3ky^2 = \frac{1}{2}k(8x^2 - 12xy + 9y^2) \tag{12.18}$$

$$\begin{cases} 2m\ddot{x} = -\partial V/\partial x, \\ 3m\ddot{y} = -\partial V/\partial y, \end{cases} \quad \text{or} \quad \begin{cases} -2m\omega^2 x = \quad -k(8x - 6y), \\ -3m\omega^2 y = -k(-6x + 9y). \end{cases} \tag{12.19}$$

Next divide each equation by its mass and write the equations in matrix form.

$$\omega^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{k}{m} \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{12.20}$$

With $\lambda = m\omega^2/k$, the eigenvalues of the square matrix are $\lambda = 1$ and $\lambda = 6$. Thus the characteristic frequencies of vibration are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{6k}{m}}. \tag{12.21}$$

The corresponding eigenvectors are:

$$\text{For } \lambda = 1\text{: } y = x \text{ or } \mathbf{r} = (1,1)\text{; for } \lambda = 6\text{: } 3y = -2x \text{ or } \mathbf{r} = (3,-2). \tag{12.22}$$

Thus at frequency $\omega_1$ the two masses oscillate back and forth together with equal amplitudes like this $\leftarrow\leftarrow$ and then like this $\rightarrow\rightarrow$. At frequency $\omega_2$ the two masses oscillate in opposite directions with amplitudes in the ratio 3 to 2 like this $\leftarrow\rightarrow$ and then like this $\rightarrow\leftarrow$.

Now we seem to have solved the problem; where is the difficulty? Note that the square matrix in (12.20) is not symmetric [and compare (12.14) where the square matrix was symmetric]. In Section 11 we discussed the fact that (for real matrices) only symmetric matrices have orthogonal eigenvectors and can be diagonalized by an orthogonal transformation. Here note that the eigenvectors in Example 3 were orthogonal [dot product of $(1,1)$ and $(1,-1)$ is zero] but the eigenvectors for (12.20) are not orthogonal [dot product of $(1,1)$ and $(3,-2)$ is not zero]. If we want orthogonal eigenvectors, we can make the change of variables (also see Example 6)

$$X = x\sqrt{2}, \quad Y = y\sqrt{3}, \tag{12.23}$$

where the constants are the square roots of the numerical factors in the masses $2m$ and $3m$. (Note that geometrically this just amounts to different changes in scale along the two axes, not to a rotation.) Then (12.20) becomes

$$\omega^2 \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{k}{m} \begin{pmatrix} 4 & -\sqrt{6} \\ -\sqrt{6} & 3 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{12.24}$$

By inspection we see that the characteristic equation for the square matrix in (12.24) is the same as the characteristic equation for (12.20) so the eigenvalues and the characteristic frequencies are the same as before (as they must be by physical reasoning). However the (12.24) matrix is symmetric and so we know that its eigenvectors are orthogonal. By direct substitution of (12.23) into (12.22), [or by solving for the eigenvectors in the (12.24) matrix] we find the eigenvectors in the $X$, $Y$ coordinates:

$$\text{For } \lambda = 1\text{: } \mathbf{R} = (X,Y) = (\sqrt{2}, -\sqrt{3})\text{; for } \lambda = 6\text{: } \mathbf{R} = (3\sqrt{2}, 2\sqrt{3}). \tag{12.25}$$

As expected, these eigenvectors are orthogonal.

▶ **Example 5.** Let's consider a model of a linear triatomic molecule in which we approximate the forces between the atoms by forces due to springs (Figure 12.2).

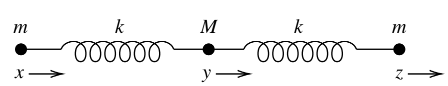


**Figure 12.2**

As in Example 3, let $x$, $y$, $z$ be the coordinates of the three masses relative to their equilibrium positions. We want to find the characteristic vibration frequencies of the molecule. Following our work in Examples 3 and 4, we find (Problem 12)

$$V = \tfrac{1}{2}k(x-y)^2 + \tfrac{1}{2}k(y-z)^2 = \tfrac{1}{2}k(x^2+2y^2+z^2-2xy-2yz), \tag{12.26}$$

$$\begin{cases} m\ddot{x} = -\partial V/\partial x = -k(x-y), \\ M\ddot{y} = -\partial V/\partial y = -k(2y-x-z), \\ m\ddot{z} = -\partial V/\partial z = -k(z-y), \end{cases}$$

(12.27) or

$$\begin{cases} -m\omega^2 x = -k(x-y), \\ -M\omega^2 y = -k(2y-x-z), \\ -m\omega^2 z = -k(z-y). \end{cases}$$

We are going to consider several different ways of solving this problem in order to learn some useful techniques. First of all, if we add the three equations we get

$$m\ddot{x} + M\ddot{y} + m\ddot{z} = 0. \tag{12.28}$$

Physically (12.28) says that the center of mass is at rest or moving at constant speed (that is, has zero acceleration). Since we are just interested in vibrational motion, let's assume that the center of mass is at rest at the origin. Then we have $mx + My + mz = 0$. Solving this equation for $y$ gives

$$y = -\frac{m}{M}(x+z). \tag{12.29}$$

Substitute (12.29) into the second set of equations in (12.27) to get the $x$ and $z$ equations

$$\begin{aligned} -m\omega^2 x &= -k(1+\frac{m}{M})x - k\frac{m}{M}z, \\ -m\omega^2 z &= -k\frac{m}{M}x - k(1+\frac{m}{M})z. \end{aligned} \tag{12.30}$$

In matrix form equations (12.30) become [compare (12.14)]

$$\lambda\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+\frac{m}{M} & \frac{m}{M} \\ \frac{m}{M} & 1+\frac{m}{M} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \lambda = \frac{m\omega^2}{k}. \tag{12.31}$$

We solve this eigenvalue problem to find

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}. \tag{12.32}$$

For $\omega_1$ we find $z = -x$, and consequently by (12.29), $y = 0$. For $\omega_2$, we find $z = x$ and so $y = -\frac{2m}{M}x$. Thus at frequency $\omega_1$, the central mass $M$ is at rest and the two masses $m$ vibrate in opposite directions like this $\leftarrow m \;\; M \;\; m\rightarrow$ and then like this $m\rightarrow \;\; M \;\; \leftarrow m$. At the higher frequency $\omega_2$, the central mass $M$ moves in one direction while the two masses $m$ move in the opposite direction, first like this $m\rightarrow \; \leftarrow M \;\; m\rightarrow$ and then like this $\leftarrow m \;\; M\rightarrow \; \leftarrow m$.

Now suppose that we had not thought about eliminating the translational motion and had set this problem up as a 3 variable problem. Let's go back to the second set of equations in (12.27), and divide the $x$ and $z$ equations by $m$ and the $y$ equation by $M$. Then in matrix form these equations can be written as

$$\omega^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{k}{m} \begin{pmatrix} 1 & -1 & 0 \\ \frac{-m}{M} & \frac{2m}{M} & \frac{-m}{M} \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{12.33}$$

With $\lambda = m\omega^2/k$, the eigenvalues of the square matrix are $\lambda = 0, 1, 1 + \frac{2m}{M}$, and the corresponding eigenvectors are (check these)

$$\begin{aligned} &\text{For } \lambda = 0,\ \mathbf{r} = (1,1,1); \\ &\text{for } \lambda = 1,\ \mathbf{r} = (1,0,-1); \\ &\text{for } \lambda = 1 + \frac{2m}{M},\ \mathbf{r} = (1, -\frac{2m}{M}, 1). \end{aligned} \tag{12.34}$$

We recognize the $\lambda = 0$ solution as corresponding to translation both because $\omega = 0$ (so there is no vibration), and because $\mathbf{r} = (1,1,1)$ says that any motion is the same for all three masses. The other two modes of vibration are the same ones we had above. We note that the square matrix in (12.33) is not symmetric and so, as expected, the eigenvectors in (12.34) are not an orthogonal set. However, the last two (which correspond to vibrations) are orthogonal so if we are just interested in modes of vibration we can ignore the translation eigenvector. If we want to consider all motion of the molecule along its axis (both translation and vibration), and want an orthogonal set of eigenvectors, we can make the change of variables discussed in Example 4, namely

$$X = x, \qquad Y = y\sqrt{\frac{M}{m}}, \qquad Z = z. \tag{12.35}$$

Then the eigenvectors become

$$(1, \sqrt{M/m}\,, 1), \quad (1,0,-1), \quad (1, -2\sqrt{m/M}\,, 1) \tag{12.36}$$

which are an orthogonal set. The first eigenvector (corresponding to translation) may seem confusing, looking as if the central mass $M$ doesn't move with the others (as it must for pure translation). But remember from Example 4 that changes of variable like (12.23) and (12.35) correspond to changes of scale, so in the $XYZ$ system we are not using the same measuring stick to find the position of the central mass as for the other two masses. Their physical displacements are actually all the same.

**Example 6.** Let's consider Example 4 again in order to illustrate a very compact form for the eigenvalue equation. Satisfy yourself (Problem 13) that we can write the potential energy $V$ in (12.18) as

$$V = \frac{1}{2}k\mathrm{r}^{\mathrm{T}}\mathrm{V}\mathrm{r} \quad \text{where} \quad V = \begin{pmatrix} 8 & -6 \\ -6 & 9 \end{pmatrix}, \quad \mathrm{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathrm{r}^{\mathrm{T}} = \begin{pmatrix} x & y \end{pmatrix}. \tag{12.37}$$

Similarly the kinetic energy $T = \frac{1}{2}(2m\dot{x}^2 + 3m\dot{y}^2)$ can be written as

$$T = \frac{1}{2}m\dot{\mathrm{r}}^{\mathrm{T}}\mathrm{T}\dot{\mathrm{r}} \quad \text{where} \quad \mathrm{T} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \dot{\mathrm{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \quad \dot{\mathrm{r}}^{\mathrm{T}} = \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}. \tag{12.38}$$

(Notice that the T matrix is diagonal and is a unit matrix when the masses are equal; otherwise T has the mass factors along the main diagonal and zeros elsewhere.) Now using the matrices T and V, we can write the equations of motion (12.19) as

$$m\omega^2 \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = k \begin{pmatrix} 8 & -6 \\ -6 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or}$$

$$\lambda \mathrm{Tr} = \mathrm{Vr} \quad \text{where} \quad \lambda = \frac{m\omega^2}{k}. \tag{12.39}$$

We can think of (12.39) as the basic eigenvalue equation. If T is a unit matrix, then we just have $\lambda \mathrm{r} = \mathrm{Vr}$ as in (12.14). If not, then we can multiply (12.39) by $\mathrm{T}^{-1}$ to get

$$\lambda \mathrm{r} = \mathrm{T}^{-1}\mathrm{Vr} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 8 & -6 \\ -6 & 9 \end{pmatrix} \mathrm{r} = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{12.40}$$

as in (12.20). However, we see that this matrix is not symmetric and so the eigenvectors will not be orthogonal. If we want the eigenvectors to be orthogonal as in (12.23), we choose new variables so that the T matrix is the unit matrix, that is variables $X$ and $Y$ so that

$$T = \tfrac{1}{2}(2m\dot{x}^2 + 3m\dot{y}^2) = \tfrac{1}{2}m(\dot{X}^2 + \dot{Y}^2). \tag{12.41}$$

But this means that we want $X^2 = 2x^2$ and $Y^2 = 3y^2$ as in (12.23), or in matrix form,

$$\mathrm{R} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x\sqrt{2} \\ y\sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{T}^{1/2}\mathrm{r} \quad \text{or}$$

$$\mathrm{r} = \mathrm{T}^{-1/2}\mathrm{R} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{12.42}$$

Substituting (12.42) into (12.39), we get $\lambda \mathrm{T}\mathrm{T}^{-1/2}\mathrm{R} = \mathrm{V}\mathrm{T}^{-1/2}\mathrm{R}$. Then multiplying on the left by $\mathrm{T}^{-1/2}$ and noting that $\mathrm{T}^{-1/2}\mathrm{T}\mathrm{T}^{-1/2} = \mathrm{I}$, we have

$$\lambda \mathrm{R} = \mathrm{T}^{-1/2}\mathrm{V}\mathrm{T}^{-1/2}\mathrm{R} \tag{12.43}$$

as the eigenvalue equation in terms of the new variables $X$ and $Y$. Substituting the numerical $\mathrm{T}^{-1/2}$ from (12.42) into (12.43) gives the result we had in (12.24).

We have simply demonstrated that (12.39) and (12.43) give compact forms of the eigenvalue equations for Example 4. However, it is straightforward to show that these equations are just a compact summary of the equations of motion for any similar vibrations problem, in any number of variables, just by writing the potential and kinetic energy matrices and comparing the equations of motion in matrix form.

▶ **Example 7.** Find the characteristic frequencies and the characteristic modes of vibration for the system of masses and springs shown in Figure 12.3, where the motion is along a vertical line.

Let's use the simplified method of Example 6 for this problem. We first write the expressions for the kinetic energy and the potential energy as in previous examples.

(Note carefully that we measure $x$ and $y$ from the equilibrium positions of the masses when they are hanging at rest; then the gravitational forces are already balanced and gravitational potential energy does not come into the expression for $V$.)

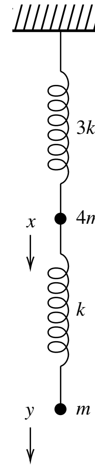


**Figure 12.3**

$$T = \tfrac{1}{2}m(4\dot{x}^2 + \dot{y}^2),$$
$$V = \tfrac{1}{2}k[3x^2 + (x-y)^2] = \tfrac{1}{2}k(4x^2 - 2xy + y^2). \tag{12.44}$$

The corresponding matrices are [see equations (12.37) and (12.38)]:

$$\mathrm{T} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{V} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}. \tag{12.45}$$

As in equation (12.40), we find $\mathrm{T}^{-1}\mathrm{V}$ and its eigenvalues and eigenvectors.

$$\mathrm{T}^{-1}\mathrm{V} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/4 \\ -1 & 1 \end{pmatrix}, \quad \lambda = \frac{m\omega^2}{k} = \frac{1}{2}, \frac{3}{2}.$$

$$\text{For } \omega = \sqrt{\frac{k}{2m}},\ \mathbf{r} = (1,2); \quad \text{for } \omega = \sqrt{\frac{3k}{2m}},\ \mathbf{r} = (1,-2). \tag{12.46}$$

As expected (since $\mathrm{T}^{-1}\mathrm{V}$ is not symmetric), the eigenvectors are not orthogonal. If we want orthogonal eigenvectors, we make the change of variables $X = 2x$, $Y = y$, to find the eigenvectors $\mathbf{R} = (1,1)$ and $\mathbf{R} = (1,-1)$ which are orthogonal. Alternatively, we can find the matrix $\mathrm{T}^{-1/2}\mathrm{V}\mathrm{T}^{-1/2}$

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \tag{12.47}$$

and find its eigenvalues and eigenvectors.

## PROBLEMS, SECTION 12

**1.** Verify that (12.2) multiplied out is (12.1).

Find the equations of the following conics and quadric surfaces relative to principal axes.

**2.** $2x^2 + 4xy - y^2 = 24$

**3.** $8x^2 + 8xy + 2y^2 = 35$

**4.** $3x^2 + 8xy - 3y^2 = 8$

**5.** $5x^2 + 3y^2 + 2z^2 + 4xz = 14$

**6.** $x^2 + y^2 + z^2 + 4xy + 2xz - 2yz = 12$

**7.** $x^2 + 3y^2 + 3z^2 + 4xy + 4xz = 60$

**8.** Carry through the details of Example 2 to find the unit eigenvectors. Show that the resulting rotation matrix C is orthogonal. *Hint:* Find $\mathrm{CC}^{\mathrm{T}}$.

**9.** For Problems 2 to 7, find the rotation matrix C which relates the principal axes and the original axes. See Example 2.

**10.** Verify equations (12.13) and (12.14). Solve (12.15) to find the eigenvalues and verify (12.16). Find the corresponding eigenvectors as stated in (12.17).

**11.** Verify the details of Example 4, equations (12.18) to (12.25).

**12.** Verify the details of Example 5, equations (12.26) to (12.36).

**13.** Verify the details of Example 6, equations (12.37) to (12.43).

Find the characteristic frequencies and the characteristic modes of vibration for systems of masses and springs as in Figure 12.1 and Examples 3, 4, and 6 for the following arrays.

**14.** $k, m, 2k, m, k$

**15.** $5k, m, 2k, m, 2k$

**16.** $4k, m, 2k, m, k$

**17.** $3k, 3m, 2k, 4m, 2k$

**18.** $2k, m, k, 5m, 10k$

**19.** $4k, 2m, k, m, k$

**20.** Carry through the details of Example 7.

Find the characteristic frequencies and the characteristic modes of vibration as in Example 7 for the following arrays of masses and springs, reading from top to bottom in a diagram like Figure 12.3.

**21.** $3k, m, 2k, m$

**22.** $4k, 3m, k, m$

**23.** $2k, 4m, k, 2m$

## 13. A BRIEF INTRODUCTION TO GROUPS

We will not go very far into group theory—there are whole books on the subject as well as on its applications in physics. But since so many of the ideas we are discussing in this chapter are involved, it is interesting to have a quick look at groups.

**Example 1.** Think about the four numbers $\pm 1$, $\pm i$. Notice that no matter what products and powers of them we compute, we never get any numbers besides these four. This property of a set of elements with a law of combination is called *closure*. Now think about these numbers written in polar form: $e^{i\pi/2}$, $e^{i\pi}$, $e^{3i\pi/2}$, $e^{2i\pi} = 1$, or the corresponding rotations of a vector (in the $xy$ plane with tail at the origin), or the set of rotation matrices corresponding to these successive 90° rotations of a vector (Problem 1). Note also that these numbers are the four fourth roots of 1, so we could write them as $A$, $A^2$, $A^3$, $A^4 = 1$. All these sets are examples of groups, or more precisely, they are all *representations* of the same group known as the *cyclic group of order* 4. We will be particularly interested in groups of matrices, that is, in matrix representations of groups, since this is very important in applications. Now just what is a group?

**Definition of a Group** A group is a set $\{A, B, C, \cdots\}$ of elements—which may be numbers, matrices, operations (such as the rotations above)—together with a law of combination of two elements (often called the "product" and written as $AB$—see discussion below) subject to the following four conditions.

1. Closure: The combination of any two elements is an element of the group.
2. Associative law: The law of combination satisfies the associative law: $(AB)C = A(BC)$.
3. Unit element: There is a unit element $I$ with the property that $IA = AI = A$ for every element of the group.

4. Inverses: Every element of the group has an inverse in the group; that is, for any element $A$ there is an element $B$ such that $AB = BA = I$.

We can easily verify that these four conditions are satisfied for the set $\pm 1$, $\pm i$ under multiplication.

1. We have already discussed closure.
2. Multiplication of numbers is associative.
3. The unit element is 1.
4. The numbers $i$ and $-i$ are inverses since their product is 1; $-1$ is its own inverse, and 1 is its own inverse.

Thus the set $\pm 1$, $\pm i$, under the operation of multiplication, is a group. The *order of a finite group* is the number of elements in the group. When the elements of a group of order $n$ are of the form $A$, $A^2$, $A^3$, $\cdots$, $A^n = 1$, it is called a *cyclic group*. Thus the group $\pm 1$, $\pm i$, under multiplication, is a cyclic group of order 4 as we claimed above.

A *subgroup* is a subset which is itself a group. The whole group, or the unit element, are called *trivial subgroups*; any other subgroup is called a *proper subgroup*. The group $\pm 1$, $\pm i$ has the proper subgroup $\pm 1$.

**Product, Multiplication Table** In the definition of a group and in the discussion so far, we have used the term "product" and have written $AB$ for the combination of two elements. However, terms like "product" or "multiplication" are used here in a generalized sense to refer to whatever the operation is for combining group elements. In applications, group elements are often matrices and the operation is matrix multiplication. In general mathematical group theory, the operation might be, for example, addition of two elements, and that sounds confusing to say "product" when we mean sum! Look at one of the first examples we discussed, namely the rotation of a vector by angles $\pi/2$, $\pi$, $3\pi/2$, $2\pi$ or 0. If the group elements are rotation matrices, then we multiply them, but if the group elements are the angles, then we add them. But the physical problem is exactly the same in both cases. So remember that group multiplication refers to the law of combination for the group rather than just to ordinary multiplication in arithmetic.

Multiplication tables for groups are very useful; equations (13.1), (13.2), and (13.4) show some examples. Look at (13.1) for the group $\pm 1$, $\pm i$. The first column and the top row (set off by lines) list the group elements. The sixteen possible products of these elements are in the body of the table. Note that each element of the group appears exactly once in each row and in each column (Problem 3). At the intersection of the row starting with $i$ and the column headed by $-i$, you find the product $(i)(-i) = 1$, and similarly for the other products.

(13.1)

| | $1$ | $i$ | $-1$ | $-i$ |
|---|---|---|---|---|
| $1$ | $1$ | $i$ | $-1$ | $-i$ |
| $i$ | $i$ | $-1$ | $-i$ | $1$ |
| $-1$ | $-1$ | $-i$ | $1$ | $i$ |
| $-i$ | $-i$ | $1$ | $i$ | $-1$ |

In (13.2) below, note that you add the angles as we discussed above. However, it's not quite just adding—it's really the familiar process of adding angles until you get to $2\pi$ and then starting over again at zero. In mathematical language this is called adding (mod $2\pi$) and we write $\pi/2 + 3\pi/2 \equiv 0 \pmod{2\pi}$. Hours on an ordinary clock add in a similar way. If it's 10 o'clock and then 4 hours elapse, the clock says it's 2 o'clock. We write $10 + 4 \equiv 2 \pmod{12}$. (See Problems 6 and 7 for more examples.)

(13.2)

| | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ |
|---|---|---|---|---|
| $0$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ |
| $\pi/2$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $0$ |
| $\pi$ | $\pi$ | $3\pi/2$ | $0$ | $\pi/2$ |
| $3\pi/2$ | $3\pi/2$ | $0$ | $\pi/2$ | $\pi$ |

Two groups are called *isomorphic* if their multiplication tables are identical except for the names we attach to the elements [compare (13.1) and (13.2)]. Thus all the 4-element groups we have discussed so far are isomorphic to each other, that is, they are really all the same group. However, there are two different groups of order 4, the cyclic group we have discussed, and another group called the 4's group (see Problem 4).

**Symmetry Group of the Equilateral Triangle** Consider three identical atoms at the corners of an equilateral triangle in the $xy$ plane, with the center of the triangle at the origin as shown in Figure 13.1. What rotations and reflections of vectors in the $xy$ plane (as in Section 7) will produce an identical array of atoms? By considering Figure 13.1, we see that there are three possible rotations: 0°, 120°, 240°, and three possible reflections, through the three lines $F$, $G$, $H$ (lines along the altitudes of the triangle). Think of moving just the triangle (that is, the atoms), leaving the axes and the lines $F$, $G$, $H$ fixed in the background. As in Section 7, we can write a 2 by 2 rotation or reflection matrix for each of these six transformations and set up a multiplication table to show that they do form a group of order 6. This group is called the symmetry group of the equilateral triangle. We find (Problem 8)

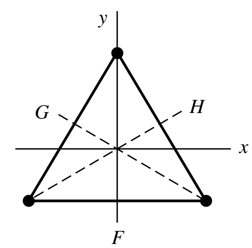


**Figure 13.1**

(13.3)

Identity, 0° rotation $\quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

120° rotation $\quad \mathrm{A} = \dfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

240° rotation $\quad \mathrm{B} = \dfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$

Reflection through line $F$ ($y$ axis) $\quad \mathrm{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection through line $G$ $\quad \mathrm{G} = \dfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$

Reflection through line $H$ $\quad \mathrm{H} = \dfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

The group multiplication table is:

(13.4)

| | I | A | B | F | G | H |
|---|---|---|---|---|---|---|
| I | I | A | B | F | G | H |
| A | A | B | I | G | H | F |
| B | B | I | A | H | F | G |
| F | F | H | G | I | B | A |
| G | G | F | H | A | I | B |
| H | H | G | F | B | A | I |

Note here that GF = A, but FG = B, not surprising since we know that matrices don't always commute. In group theory, if every two group elements commute, the group is called *Abelian*. Our previous group examples have all been Abelian, but the group in (13.4) is not Abelian.

This is just one example of a symmetry group. Group theory is so important in applications because it offers a systematic way of using the symmetry of a physical problem to simplify the solution. As we have seen, groups can be represented by sets of matrices, and this is widely used in applications.

**Conjugate Elements, Class, Character** Two group elements A and B are called *conjugate* elements if there is a group element C such that $C^{-1}AC = B$. By letting C be successively one group element after another, we can find all the group elements conjugate to A. This set of conjugate elements is called a *class*. Recall from Section 11 that if A is a matrix describing a transformation (such as a rotation or some sort of mapping of a space onto itself), then $B = C^{-1}AC$ describes the same mapping but relative to a different set of axes (different basis). Thus all the elements of a class really describe the same mapping, just relative to different bases.

▶ **Example 2.** Find the classes for the group in (13.3) and (13.4). We find the elements conjugate to F as follows [use (13.4) to find inverses and products]:

$$\begin{aligned} I^{-1}FI &= F; \\ A^{-1}FA &= BFA = BH = G; \\ B^{-1}FB &= AFB = AG = H; \\ F^{-1}FF &= F; \\ G^{-1}FG &= GFG = GB = H; \\ H^{-1}FH &= HFH = HA = G. \end{aligned} \tag{13.5}$$

Thus the elements F, G, and H are conjugate to each other and form one class. You can easily show (Problem 12) that elements A and B are another class, and the unit element I is a class by itself. Now notice what we observed above. The elements F, G, and H all just interchange two atoms, that is, all of them do the same thing, just seen from a different viewpoint. The elements A and B rotate the atoms, A by 120° and B by 240° which is the same as 120° looked at upside down. And finally the unit element I leaves things unchanged so it is a class by itself. Notice that a class is not a group (except for the class consisting of I) since a group must contain the unit element. So a class is a subset of a group, but not a subgroup.

Recall from (9.13) and Problem 11.10 that the trace of a matrix (sum of diagonal elements) is not changed by a similarity transformation. Thus all the matrices of a class have the same trace. Observe that this is true for the group (13.3): Matrix I has trace = 2, A and B have trace = $-\frac{1}{2} - \frac{1}{2} = -1$, and F, G, and H have trace = 0. In this connection, the trace of a matrix is called its *character*, so we see that all matrices of a class have the same character. Also note that we could write the matrices (13.3) in (infinitely) many other ways by rotating the reference axes, that is, by performing similarity transformations. But since similarity transformations do not change the trace, that is, the character, we now have a number attached to each class which is independent of the particular choice of coordinate system (basis). Classes and their associated character are very important in applications of group theory.

One more number is important here, and that is the dimension of a representation. In (13.3), we used 2 by 2 matrices (2 dimensions), but it would be possible to work in 3 dimensions. Then, for example, the A matrix would describe a 120° rotation around the $z$ axis and would be

$$\mathrm{A} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{13.6}$$

and the other matrices in (13.3) would have a similar form, called *block diagonalized*. But now the traces of all the matrices are increased by 1. To avoid having any ambiguity about character, we use what are called "irreducible representations" in finding character; let's discuss this.

**Irreducible Representations** A 2-dimensional representation is called *reducible* if all the group matrices can be diagonalized by the same unitary similarity transformation (that is, the same change of basis). For example, the matrices in Problem 1 and the matrices in Problem 4 both give 2-dimensional reducible representations of their groups (see Problems 13, 15, and 16). On the other hand, the matrices in (13.3) cannot be simultaneously diagonalized (see Problem 13), so (13.3) is called a 2-dimensional *irreducible representation* of the equilateral triangle symmetry group. If a group of 3 by 3 matrices can all be either diagonalized or put in the form of (13.6) (block diagonalized) by the same unitary similarity transformation, then the representation is called reducible; if not, it is a 3-dimensional irreducible representation. For still larger matrices, imagine the matrices block diagonalized with blocks along the main diagonal which are the matrices of irreducible representations.

Thus we see that any representation is made up of irreducible representations. For each irreducible representation, we find the character of each class. Such lists are known as character tables, but their construction is beyond our scope.

**Infinite Groups** Here we survey some examples of infinite groups as well as some sets which are not groups.

(13.7)

(a) The set of all integers, positive, negative, and zero, under ordinary addition, is a group. *Proof*: The sum of two integers is an integer. Ordinary addition obeys the associative law. The unit element is 0. The inverse of the integer N is −N since $\mathrm{N} + (-\mathrm{N}) = 0$.

(b) The same set under ordinary multiplication is not a group because 0 has no inverse. But even if we omit 0, the inverses of the other integers are fractions which are not in the set.

(c) Under ordinary multiplication, the set of all rational numbers except zero, is a group. *Proof*: The product of two rational numbers is a rational number. Ordinary multiplication is associative. The unit element is 1, and the inverse of a rational number is just its reciprocal.

Similarly, you can show that the following sets are groups under ordinary multiplication (Problem 17): All real numbers except zero, all complex numbers except zero, all complex numbers $re^{i\theta}$ with $r = 1$.

(d) Ordinary subtraction or division cannot be group operations because they don't satisfy the associative law; for example, $x-(y-z) \neq (x-y)-z$. (Problem 18.)

(e) The set of all orthogonal 2 by 2 matrices under matrix multiplication is a group called O(2). If the matrices are required to be rotation matrices, that is, have determinant +1, the set is a group called SO(2) (the S stands for special). Similarly, the following sets of matrices are groups under matrix multiplication: The set of all orthogonal 3 by 3 matrices, called O(3); its subgroup SO(3) with determinant = 1; or the corresponding sets of orthogonal matrices of any dimension $n$, called O($n$) and SO($n$). (Problem 19.)

(f) The set of all unitary $n$ by $n$ matrices, $n = 1, 2, 3, \cdots$, called U($n$), is a group under matrix multiplication, and its subgroup SU($n$) of unitary matrices with determinant = 1 is also a group. *Proof*: We have repeatedly noted that matrix multiplication is associative and that the unit matrix is the unit element of a group of matrices. So we just need to check closure and inverses. The product of two unitary matrices is unitary (see Section 9). If two matrices have determinant = 1, their product has determinant = 1 [see equation (6.6)]. The inverse of a unitary matrix is unitary (see Problem 9.25).

## ▶ PROBLEMS, SECTION 13

**1.** Write the four rotation matrices for rotations of vectors in the $xy$ plane through angles 90°, 180°, 270°, 360° (or 0°) [see equation (7.12)]. Verify that these 4 matrices under matrix multiplication satisfy the four group requirements and are a matrix representation of the cyclic group of order 4. Write their multiplication table and compare with Equations (13.1) and (13.2).

**2.** Following the text discussion of the cyclic group of order 4, and Problem 1, discuss

(a) the cyclic group of order 3 (see Chapter 2, Problem 10.32);

(b) the cyclic group of order 6.

**3.** Show that, in a group multiplication table, each element appears exactly once in each row and in each column. *Hint:* Suppose that an element appears twice, and show that this leads to a contradiction, namely that two elements assumed different are the same element.

**4.** Show that the matrices

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

| (9.1) | *Name of Matrix* | *Notations for it* | *How to get it from* A |
|---|---|---|---|
| | Transpose of A, or A transpose | $A^T$ or $\widetilde{A}$ or $A'$ or $A^t$ | Interchange rows and columns in A. |
| | Complex conjugate of A | $\bar{A}$ or $A^*$ | Take the complex conjugate of each element. |
| | Transpose conjugate, Hermitian conjugate, adjoint (Problem 9), Hermitian adjoint. | $A^\dagger$ (A dagger) | Take the complex conjugate of each element and transpose. |
| | Inverse of A | $A^{-1}$ | See Formula (6.13). |

| (9.2) | A matrix is called | if it satisfies the condition(s) |
|---|---|---|
| | real | $A = \bar{A}$ |
| | symmetric | $A = A^T$, A real<br>(matrix = its transpose) |
| | skew-symmetric or antisymmetric | $A = -A^T$, A real |
| | orthogonal | $A^{-1} = A^T$, A real<br>(inverse = transpose) |
| | pure imaginary | $A = -\bar{A}$ |
| | Hermitian | $A = A^\dagger$<br>(matrix = its transpose conjugate) |
| | anti-Hermitian | $A = -A^\dagger$ |
| | unitary | $A^{-1} = A^\dagger$<br>(inverse = transpose conjugate) |
| | normal | $AA^\dagger = A^\dagger A$<br>(A and $A^\dagger$ commute) |