

Convolutions:

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$$F \otimes g$$

Additive Version

$$= \int dx \int dy F(x) g(y) \delta[a - (x+y)]$$

$$= \int dx F(x) g(a-x) \quad w/ \quad y = a-x$$

$$= \int dy F(a-y) g(y) \quad w/ \quad x = a-y$$

Convolution is symmetric!!!

Multiplicative Version

$$F \otimes g =$$

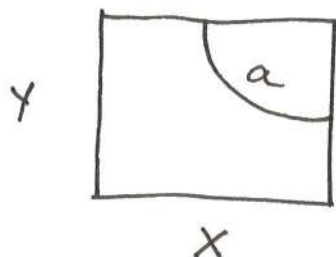
$$= \int dx \int dy F(x) g(y) \delta[a - xy]$$

$$= \int \frac{dx}{x} F(x) g\left(\frac{a}{x}\right) \quad w/ \quad y = \frac{a}{x}$$

$$= \int \frac{dy}{y} F\left(\frac{a}{y}\right) g(y) \quad w/ \quad x = \frac{a}{y}$$

[Special Case] F, g defined $\in [0, 1]$ $a, x, y \in [0, 1]$

$$y = \frac{a}{x} \quad \text{then} \Rightarrow x \in [a, 1]$$

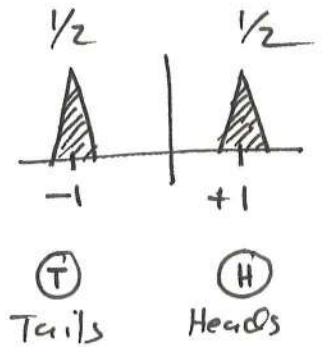


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$$F \otimes g = \int_a^1 \frac{dx}{x} F(x) g\left(\frac{a}{x}\right)$$

Example: Coin Flip:

$$F(x) = g(y) = \frac{1}{2} [\delta(x-1) + \delta(x+1)] =$$



Double Coin Flip: $F \oplus g$

$$F \oplus g = \int dx \int dy F(x) g(y) \delta[a - (x+y)]$$

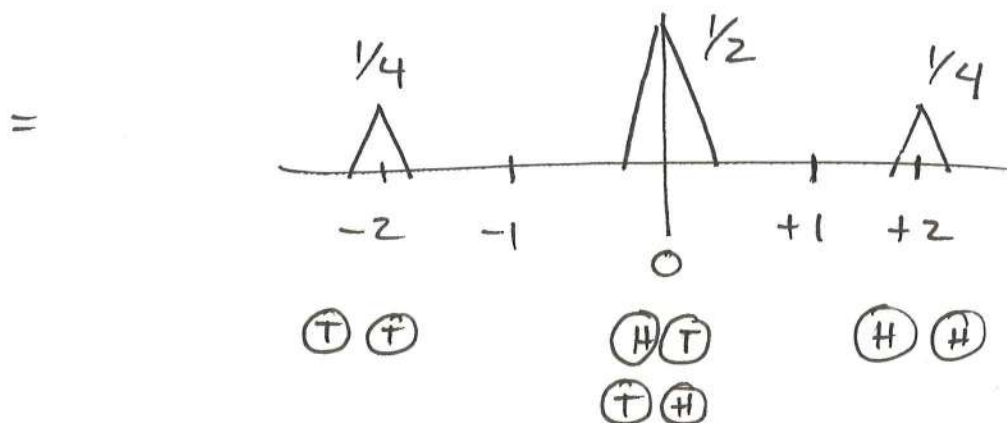
$$= \int dx F(x) g(a-x)$$

$$= \int dx \frac{1}{2} [\delta(x-1) + \delta(x+1)] \cdot \frac{1}{2} [\delta(a-x-1) + \delta(a-x+1)]$$

$$= \frac{1}{4} \int dx \left\{ \delta(x-1) [\delta(a-x-1) + \delta(a-x+1)] + \delta(x+1) [\delta(a-x-1) + \delta(a-x+1)] \right\}$$

$$= \frac{1}{4} \left\{ \delta(a-2) + \delta(a) + \delta(a) + \delta(a+2) \right\}$$

$$= \frac{1}{4} \left\{ \delta(a-2) + 2\delta(a) + \delta(a+2) \right\}$$



Example 2:Compute average value of $F(x) = \frac{1}{2} [\delta(x-1) + \delta(x+1)]$

$$\begin{aligned} \int dx \ x F(x) &= \int dx \ x \cdot \frac{1}{2} [\delta(x-1) + \delta(x+1)] \\ &= \cancel{\int dx} \ \frac{1}{2} [(+1) + (-1)] = 0 \end{aligned}$$

Compute average squared $\overline{x^2}$:

$$\begin{aligned} \int dx \ x^2 F(x) &= \int dx \ x^2 \cdot \frac{1}{2} [\delta(x-1) + \delta(x+1)] \\ \cancel{\int dx} &= \int dx \cdot \frac{1}{2} [(+1)^2 + (-1)^2] = 1 \end{aligned}$$

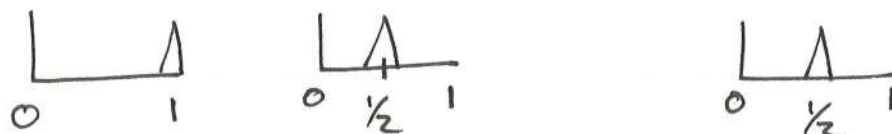
Compute $\overline{x^2}$ of $F \otimes F = \frac{1}{4} \{ \delta(x-2) + 2\delta(x) + \delta(x+2) \}$

$$\begin{aligned} \int dx \ x^2 (F \otimes F) &= \int dx \ x^2 \cdot \frac{1}{4} \{ \delta(x-2) + 2\delta(x) + \delta(x+2) \} \\ &= \int dx \ \frac{1}{4} \{ (+2)^2 + 2(0)^2 + (-2)^2 \} \\ &= \frac{8}{4} = 2 \end{aligned}$$

$$\langle \overline{x^2} \rangle = \langle x^2 \rangle = 2$$

Example

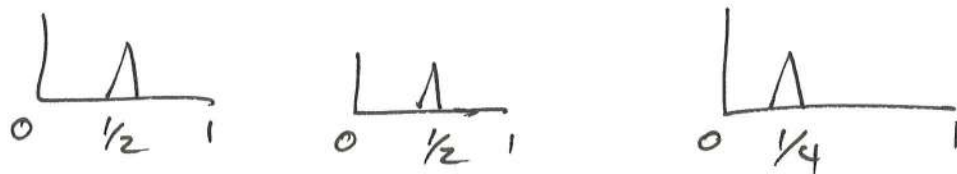
$$F \otimes g = \delta(1-x) \otimes \delta(y-\frac{1}{2}) = \delta(a-\frac{1}{2})$$



$$= \int_{\mathbb{R}} \frac{dx}{x} F(x) g\left(\frac{a}{x}\right) = \int_{\mathbb{R}} \frac{dx}{x} \delta(1-x) \delta\left(\frac{a}{x}-\frac{1}{2}\right)$$

$$= \delta(a-\frac{1}{2})$$

$$g \otimes g = \delta(x-\frac{1}{2}) \otimes \delta(y-\frac{1}{2}) = \delta(a-\frac{1}{4})$$



$$= \int \frac{dx}{x} F(x) g\left(\frac{a}{x}\right) = \int \frac{dx}{x} \delta(x-\frac{1}{2}) \delta\left(\frac{a}{x}-\frac{1}{2}\right)$$

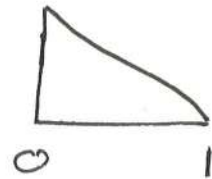
$$= 2 \delta(2a-\frac{1}{2}) = \delta(a-\frac{1}{4})$$

What do you expect for $g \otimes g \otimes g$

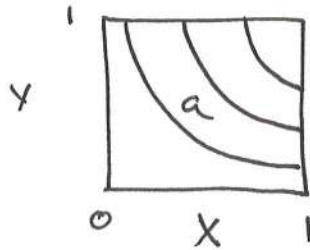
Example:

$$F \otimes F = ?$$

$$F(x) = (1-x)$$



$$F \otimes F = \int_0^1 \int_0^1 dx dy F(x) F(y) \delta(a - xy)$$



$$y = \frac{a}{x}$$

Limits on $x: \in [a, 1]$

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$$F \otimes F = \int_a^1 \frac{dx}{x} F(x) F\left(\frac{a}{x}\right) =$$

$$= \int_a^1 \frac{dx}{x} (1-x) \left(1 - \frac{a}{x}\right) =$$