Electrostatics & $I = 0$

$\nabla \cdot E = \frac{Q}{\varepsilon_0} \equiv \frac{\rho}{\varepsilon_0}$ $\qquad \nabla \times E = 0$ no current

$E = -\nabla \phi \quad \Rightarrow \quad \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$

Cartesian 2-D

$\nabla^2 = \partial_x^2 + \partial_y^2$

$\phi(x,y) = \phi(x)\,\phi(y) \equiv X(x)\,Y(y)$

$\nabla^2 \phi = (\partial_x^2 X)\, Y + X\, (\partial_y^2 Y) = 0$

$\Rightarrow \quad \frac{1}{X}\partial_x^2 X + \frac{1}{Y}\partial_y^2 Y = 0$

$\frac{1}{X}\partial_x^2 X = -\alpha^2 = -\frac{1}{Y}\partial^2 Y$

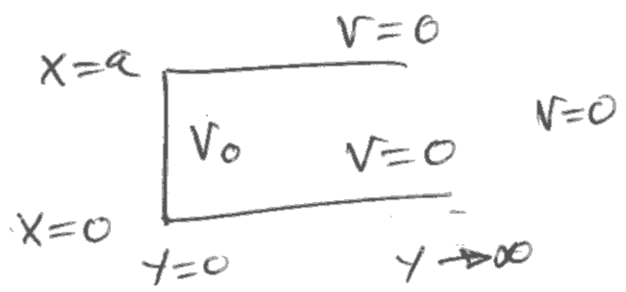
$X = e^{\pm i\alpha x}$ $\qquad Y = e^{\pm \alpha y}$

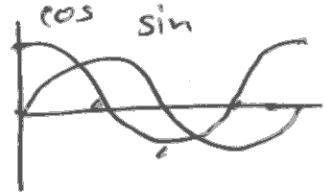
$X'' = -\alpha^2 X$ $\qquad Y'' = +\alpha^2 Y$

$e^{\pm i\alpha x} = \cos(\alpha x) \pm i \sin(\alpha x)$

$e^{\pm \alpha y} = \cosh(\alpha y) \pm \sin(\alpha y)$

Semi Infinite Groove





$$\phi(x,y) = \sum_\alpha A_\alpha \, e^{i\alpha x} \, e^{-\alpha y}$$

$$= \sum_n A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

Impose BC: $\phi(x) = 0$ at $x = [0, a]$

$\therefore$ $\alpha = \frac{n\pi}{a}$ where $n = \text{integer}$

Check y: $y \to 0$ $e^{-\frac{n\pi y}{a}} \to 1$; $y \to \infty$, $e^{()} \to 0$

Compute A_ns at $y=0$ $e^{()} = 1$

$$\phi(x,0) = \sum_n A_n \sin\left(\frac{n\pi x}{a}\right)$$

$$\int_0^a dx \, \phi(x,0) * \sin\left(\frac{m\pi x}{a}\right) = \int_0^a dx \sum_n A_n \underbrace{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right)}$$

$$\frac{2aV_0}{m\pi} \quad \boxed{\text{For } m \text{ odd}} \qquad = \delta_{mn} \frac{1}{2} A_n \, a$$

$\therefore \quad A_m = \dfrac{4V_0}{m\pi}$ for odd m

$\therefore$

$$\phi(x,y) = \sum_m \left(\frac{4V_0}{m\pi}\right) \sin\left(\frac{m\pi x}{a}\right) e^{\frac{-m\pi y}{a}}$$

Check BC $\therefore$

$x = 0 \qquad \sin(x) = 0 \quad \Rightarrow \phi = 0$

$x = a \qquad \sin(m\pi) = 0 \quad \Rightarrow \phi = 0$

$y \to \infty \qquad e^{-y} \to 0 \quad \Rightarrow \phi = 0$

For $y = 0 \quad \sum_m \left(\frac{4V_0}{m\pi}\right) \sin\left(\frac{m\pi x}{a}\right)$

Graph and check

Spherical Coord's

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

Let $\phi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$\frac{r^2}{RY}\nabla^2\phi = \frac{r^2 Y}{RY}\left[\frac{2R'}{r} + R''\right] + \frac{R\,r^2}{r^2\sin\theta}\frac{\left[\cos\theta\,Y'_\theta + \sin\theta\,Y''_\theta\right]}{RY} + \frac{R\,r^2}{r^2\sin^2\theta}\frac{Y''_\phi}{RY}$$

$$\frac{r^2}{RY}\nabla^2\phi = \left[\underbrace{2r\frac{R'}{R} + r^2\frac{R''}{R}}_{=K}\right] + \left[\frac{\cos\theta\,Y'_\theta + \sin\theta\,Y''_\theta}{\sin\theta\,Y}\right] + \left[\frac{Y''_\phi}{\sin^2\theta\,Y}\right]$$

$$\boxed{r^2R'' + 2rR' = KR}$$