

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
($x_1, x_2, x_3 = x, y, z$)

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3} \\ \nabla \cdot \mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}\end{aligned}$$

Cylindrical
(ρ, ϕ, z)

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z} \\ \nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left(\frac{1}{\rho} \frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} (\rho A_2) - \frac{\partial A_1}{\partial\phi} \right) \\ \nabla^2\psi &= \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$

Spherical
(r, θ, ϕ)

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_2) + \frac{1}{r \sin\theta} \frac{\partial A_3}{\partial\phi} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (\sin\theta A_3) - \frac{\partial A_2}{\partial\phi} \right] \\ &\quad + \mathbf{e}_2 \left[\frac{1}{r \sin\theta} \frac{\partial A_1}{\partial\phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + \mathbf{e}_3 \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial\theta} \right] \\ \nabla^2\psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \\ &\quad \left[\text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi). \right]\end{aligned}$$