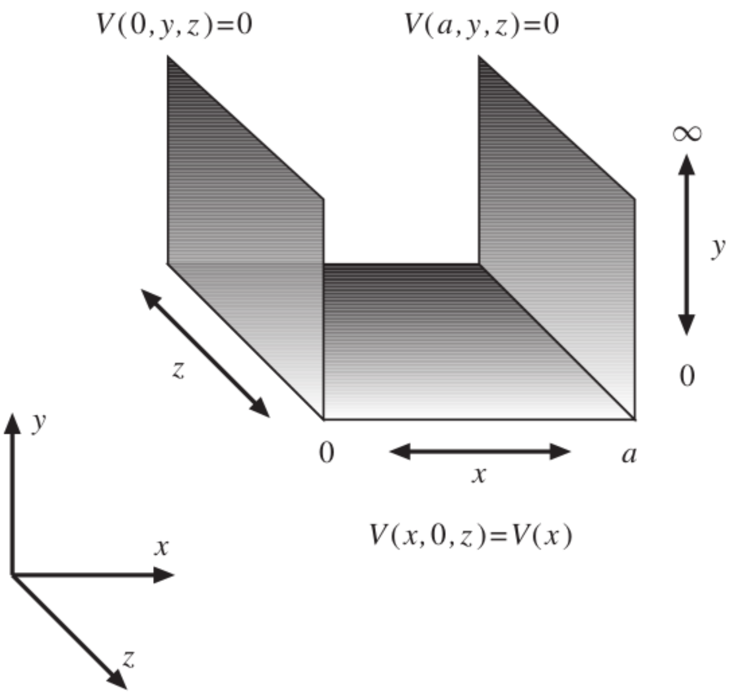
PROBLEM 2 Potential in a Rectangular Groove. Consider a rectangular groove that runs from $z = -\infty$ to $+\infty$ and is open in the positive y-direction. The groove is bounded by two parallel walls at $x = 0$, $x = a$, and at the $y = 0$ end. The walls at $x = 0$ and $x = a$ are at zero potential and the end is at a specified potential $V[x]$ that is independent of the z-coordinate. The boundary conditions suggest that the potential is independent of z. Express the potential as a superposition of products $X[x]Y[y]$, where $X[x] = \{\text{Sin}[\alpha x], \text{Cos}[\alpha x]\}$ and $Y[y] = \{e^{+\gamma y}, e^{-\gamma y}\}$. The relation between the separation constants is $\alpha^2 = \gamma^2$.

a. Express the expansion coefficients as integrals over the potential $X[x]$ given on the $y = 0$ end.



b. Let $X[x] = \text{Sin}[x]$ and write the first few terms of the potential. Plot the potential using the command **`Plot3D`**. Use the user-defined command **`VEPlot`** to plot the equipotential and electric field lines.

c. Let $X[x] = V0$ and sum the series to get an exact solution. Plot the potential using **`Plot3D`**.

d). Consider the case where the potential on the end of the groove is **`V[x]=Sin[x]`**.

7. Solve Problem 2 if the sides $x = 0$ and $x = 1$ are insulated.

19. A long conducting cylinder is placed parallel to the z axis in an originally uniform electric field in the negative x direction. The cylinder is held at zero potential. Find the potential in the region outside the cylinder. *Hints:* See Problem 7.13. You want solutions of Laplace's equation in polar coordinates (Problem 5.12).

##) For the Laplacian equation in Cartesian coordinates in 3-D: {x,y,z}

- separate the variables to obtain the differential equations for the 3 coordinates.
- Find the function that satisfies each of the differential equations.

##) For the Laplacian equation in Polar coordinates in 3-D: {r,z,ϕ }

- separate the variables to obtain the differential equations for the 3 coordinates.
- Find the function that satisfies each of the differential equations.

##) For the Laplacian equation in Spherical coordinates in 3-D: {r,θ ,ϕ }

- separate the variables to obtain the differential equations for the 3 coordinates.
- Find the function that satisfies each of the differential equations.

2.9 Separation of Variables; Laplace Equation in Rectangular Coordinates

The partial differential equations of mathematical physics are often solved conveniently by a method called *separation of variables*. In the process, one often generates orthogonal sets of functions that are useful in their own right. Equations involving the three-dimensional Laplacian operator are known to be separable in eleven different coordinate systems (see *Morse and Feshbach*, pp. 509, 655). We discuss only three of these in any detail—rectangular, spherical, and cylindrical—beginning with the simplest, rectangular coordinates.

The Laplace equation in rectangular coordinates is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2.48}$$

A solution of this *partial* differential equation can be found in terms of three *ordinary* differential equations, all of the same form, by the assumption that the potential can be represented by a product of three functions, one for each coordinate:

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \tag{2.49}$$

Substitution into (2.48) and division of the result by (2.49) yields

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0 \tag{2.50}$$

where total derivatives have replaced partial derivatives, since each term involves a function of one variable only. If (2.50) is to hold for arbitrary values of the independent coordinates, each of the three terms must be separately constant:

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -\alpha^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= -\beta^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= \gamma^2 \end{aligned}\right\} \tag{2.51}$$

where

$$\alpha^2 + \beta^2 = \gamma^2$$

If we arbitrarily choose α^2 and β^2 to be positive, then the solutions of the three ordinary differential equations (2.51) are $e^{\pm i\alpha x}$, $e^{\pm i\beta y}$, $e^{\pm\sqrt{\alpha^2+\beta^2}z}$. The potential (2.49) can thus be built up from the product solutions:

$$\Phi = e^{\pm i\alpha x}e^{\pm i\beta y}e^{\pm\sqrt{\alpha^2+\beta^2}z} \tag{2.52}$$

3.7 *Laplace Equation in Cylindrical Coordinates; Bessel Functions*

In cylindrical coordinates (ρ, ϕ, z), as shown in Fig. 3.8, the Laplace equation takes the form:

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3.71}$$

The separation of variables is accomplished by the substitution:

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z) \tag{3.72}$$

In the usual way this leads to the three ordinary differential equations:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \tag{3.73}$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \tag{3.74}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \tag{3.75}$$

The solutions of the first two equations are elementary:

$$\begin{aligned} Z(z) &= e^{\pm kz} \\ Q(\phi) &= e^{\pm i\nu\phi} \end{aligned} \tag{3.76}$$

where $\nu = m$ is an integer and k is a constant to be determined. The radial factor is

$$R(\rho) = CJ_m(k\rho) + DN_m(k\rho)$$

If the potential is finite at $\rho = 0$, $D = 0$. The requirement that the potential vanish at $\rho = a$ means that k can take on only those special values:

$$k_{mn} = \frac{x_{mn}}{a} \qquad (n = 1, 2, 3, \ldots)$$

where x_{mn} are the roots of $J_m(x_{mn}) = 0$.

Combining all these conditions, we find that the general form of the solution is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi) \tag{3.105a}$$

At $z = L$, we are given the potential as $V(\rho, \phi)$. Therefore we have

Laplace's spherical harmonics [edit]

Main article: Spherical harmonics § Laplace's spherical harmonics

Laplace's equation in spherical coordinates is:[4]

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} = 0.$$

Consider the problem of finding solutions of the form $f(r, \theta, \varphi) = R(r)\, Y(\theta, \varphi)$. By separation of variables, two differential equations result by imposing Laplace's equation:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \qquad \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} = -\lambda.$$

The second equation can be simplified under the assumption that Y has the form $Y(\theta, \varphi) = \Theta(\theta)\,\Phi(\varphi)$. Applying separation of variables again to the second equation gives way to the pair of differential equations

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$$

$$\lambda\sin^2\theta + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

Now set

$$\mu = \cos\theta, \tag{12-11}$$

remembering that, for any function $f(\mu)$,

$$\frac{df}{d\theta} = \frac{df}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{df}{d\mu} = -(1-\mu^2)^{1/2}\frac{df}{d\mu}. \tag{12-12}$$

Then the Θ equation becomes *Legendre's equation*:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0. \tag{12-13}$$

When n is an integer, its solutions are the Legendre polynomials of Sec.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$

$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$