Mechanics Overview

Relativity:

$$E^2 = p^2 + m^2$$

Note, I use the convention that $c = 1$.

Classical:

$$E = \sqrt{m^2 + p^2} = m\sqrt{1 + \frac{p^2}{m^2}} \simeq m + \frac{p^2}{2m} = m + \frac{1}{2}mv^2$$

In classical mechanics, the energy is equal to approximately the rest mass m and the classical kinetic energy $\frac{1}{2}mv^2$. Note, if there is a potential V, then the kinetic energy comes from the replacement: $E \to E - V$.

Quantum Mechanics Wave Functions:

Start from an assumed wave equation:

$$\psi(x,t) = \exp\left[+i\,\frac{p \cdot x}{\hbar} - i\,\frac{E\,t}{\hbar}\right] = \exp\left[+i\,k \cdot x - i\,\omega\,t\right]$$

where $p = \hbar\,k$ and $E = \hbar\,\omega$; note, the exponent must be dimensionless, so $p \cdot x$ and $E\,t$ have dimensions of $\hbar$.

Taking derivatives of the wave function, we have:

$$\partial_x \psi(x,t) = +i\frac{p}{\hbar}\psi(x,t) \qquad \text{thus} \qquad p = -i\hbar\partial_x = -i\hbar\nabla$$

$$\partial_t \psi(x,t) = -i\frac{E}{\hbar}\psi(x,t) \qquad \text{thus} \qquad E = +i\hbar\partial_t$$

Caution, the above definitions depend on the signs chosen for $\psi(x,t)$; we'll comment below in detail. To summarize:

$$\boxed{p = -i\hbar\nabla} \qquad \boxed{E = +i\hbar\partial_t}$$

Schrődinger Equation:

Spelling: the [oe] can be replaced with the German [ő].

The Schroedinger equation is simply the classical equation ($E = \frac{1}{2}mv^2$) extended to wave mechanics. Starting from:

$$E - V = \frac{p^2}{2\,m} \qquad \text{or equivalently} \qquad E = \frac{p^2}{2\,m} + V$$

and multiplying through by $\psi(x,t)$, we have:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\,m}\nabla^2\psi(x,t) + V(x,t)\,\psi(x,t) \ .$$

Note here that because the momentum term p^2 is quadratic, either sign of the exponential term in the wave function $\psi(x,t) \sim \exp[\pm ip\cdot x/\hbar]$ will satisfy the equation; in practice, these two solutions represent left-moving and right-moving waves. Also note that because the energy term (E) is linear, the relative sign here is important.

Klein-Gordon Equation:

We saw above that the Schroedinger equation used the classical equation (E$=\frac{1}{2}$mv^2) which is an approximation of the exact relavistic equation:

$$E^2 = p^2 + m^2 \qquad \text{or equivalently} \qquad E^2 - p^2 = m^2$$

Thus, extending this to the wave mechanics and multiplying through by $\psi(x,t)$, we have:

$$-\hbar^2\partial_t^2\psi(x,t) + \hbar^2\nabla^2\psi(x,t) = m^2\psi(x,t) \ .$$

The challenge to interpreting the Klein-Gordon equation is the existence of negative energy states. Starting from $E^2 = p^2 + m^2$, we thus have $E = \pm\sqrt{p^2 + m^2}$. The solution (over simplified) is to interpret the Klein-Gordon equation as describing spin-zero particles where the negative energy states are anti-particle states.

Dirac Equation:

The problem with the Klein-Gordon equation was the negative energy states arising from the fact that the starting equation was quadratic in energy E. Thus, Dirac attempted to obtain a linear equation. Starting from $m^2 = E^2 - p^2 = (E-p)(E+p)$, we might guess a linear equation could be:

$$m = E \pm p$$

or extending this to the wave function form:

$$\left[m = i\hbar\partial_t \mp i\hbar\nabla\right]\psi(x,t) \qquad \text{[schematic only]}$$

The above equation is only schematic, and to make it precise we can insert matrices $\{\alpha,\beta\}$ such that $[m/(i\hbar) = \beta\partial_t \mp \alpha\cdot\nabla]\psi(x,t)$, and we can relate $\{\alpha,\beta\}$ to the usual Dirac matrices $\gamma^\mu = \{\beta, \beta\alpha\}$. This yields:

$$(i\gamma^\mu\partial_\mu - m)\psi(x,t) = 0$$

where γ^μ are the 4×4 anti-commuting Dirac matrices.

We can recover the Klein-Gordon equation essentially by squaring the Dirac equation:

$$\begin{aligned} m^2 &= (E-p)(E+p) = (i\hbar\partial_t - i\hbar\nabla)(i\hbar\partial_t + i\hbar\nabla) \\ &= -\hbar^2\partial_t^2 + \hbar^2\nabla^2 \ \longrightarrow\ E^2 + p^2 \end{aligned}$$

The Dirac equation describes spin-1/2 particles and anti-particles.