1 Mellin Transform Example:

1.1 Mellin Transform Definition:

Following the Wikipedia convention:

$$\begin{aligned} \widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, f(x) \\ f(x) &= \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} dn\ x^{-n}\ \widetilde{f}(n) \end{aligned}$$

Here, $f(x)$ is the function in x-space,and $\widetilde{f}(n)$ is the Mellin transform in n-space.

1.2 Cauchy's integral formula:

We will make use of Cauchy's integral formula:

$$g(a) = \frac{1}{2\pi i}\oint \frac{g(z)}{z-a}\, dz$$

1.3 Single term: :$f(x) = x^3$.

Let's start with just a single term:$f(x) = x^3$.

$$\begin{aligned} \widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, x^3 \\ &= \int_0^1 dx\, x^{n+2} \\ &= \frac{1}{n+3} \end{aligned}$$

Now let's take the inverse Mellin transform. (Note the pole/singularity is at a negative integer.)

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} dn\ x^{-n}\ \widetilde{f}(n) \\ &= \frac{1}{2\pi i}\oint \frac{g(n)}{n-(-3)}\, dn \\ &= g(-3) = x^{-(-3)} = x^3 \end{aligned}$$

where $g(n) = x^{-n}$.

1.4 General term: :$f(x) = x^a$.

For a general term:$f(x) = x^a$.

$$\begin{aligned}
\widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, x^a \\
&= \int_0^1 dx\, x^{n+a-1} \\
&= \frac{1}{n+a}
\end{aligned}$$

Now let's take the inverse Mellin transform. (Note, we'll assume $a > 0$ so that the pole/singularity is at a negative integer.)

$$\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} dn\ x^{-n}\ \widetilde{f}(n) \\
&= \frac{1}{2\pi i}\oint \frac{g(n)}{n-(-a)}\, dn \\
&= g(-3) = x^{-(-a)} = x^a
\end{aligned}$$

where $g(n) = x^{-n}$.

1.5 General Polynomial

We can apply to a general polynomial:

$$\begin{aligned}
f(x) &= \sum_k c_k\, x^k \\
\widetilde{f}(n) &= \sum_k \frac{c_k}{n+k}
\end{aligned}$$

1.6 Factorize convolution

Consider the convolution below; this is a function of z, as x and y are integrated over.

$$[f \otimes g]\,(z) = \int dx \int dy\, f(x)\, g(y)\, \delta(z - x\, y)$$

Let's take the Mellin transform

$$\int z^{n-1}\, [f \otimes g]\,(z) = \int dz\, z^{n-1} \int dx \int dy\, f(x)\, g(y)\, \delta(z - x\, y)$$

$$
\begin{aligned}
&= \left[\int dx\, x^{n-1}\, f(x)\right] \left[\int dy\, y^{n-1}\, g(y)\right] \left[\int dz\, \delta(z - x\, y)\right] \\
&= \widetilde{f}(n)\, \widetilde{g}(n)
\end{aligned}
$$

Here, we have used the delta function to replace $z^{n-1} \to x^{n-1} y^{n-1}$, and then the delta function eliminates the $\int dz$ integral. Thus, we see that in Mellin n-space, the convolution becomes a simple production of the Mellin moments.