Complex Variables Ch 6

Motivation:

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1} = \pm i \qquad \text{where } i^2 = -1$$

Caution: $i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{+1} = 1 = -1$???

$$z = a + ib \qquad a = \mathrm{Re}[z]$$

$$b = \mathrm{Im}[z]$$

$$z_1 + z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 * z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Conjugate: $z^* = (a + ib)^* = (a - ib)$

$$i \rightarrow -i$$

$$|z| = \sqrt{z z^*} = \sqrt{(a + ib)(a + ib)^*}$$

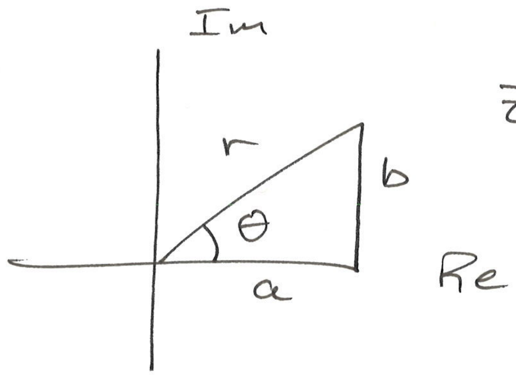
$$= \sqrt{(a + ib)(a - ib)}$$

$$= \sqrt{a^2 + b^2} \geqslant 0$$

Division:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{z^*}{z^*} = \frac{z^*}{|z|^2} = \frac{a - ib}{a^2 + b^2}$$

Polar Representation:



$$z = a + ib = r e^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$

$a = r\cos\theta$ $\qquad$ $r = \sqrt{a^2 + b^2}$

$b = r\sin\theta$ $\qquad$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Note: $\quad e^{i\theta} = \cos\theta + i\sin\theta$

(Proof is similar to $e^{i\theta\sigma_z} = \mathbb{I}\cos\theta + i\sigma_z\sin\theta$)

De Moivre's Formula:

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n \equiv \cos(n\theta) + i\sin(n\theta)$$

Special Case: $n = 2$

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin(2\theta)$$

$$(\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) =$$

$$\Rightarrow \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta \; ; \; \sin(2\theta) = 2\cos\theta\sin\theta$$

$$z = r e^{i\theta} \qquad z^* = r e^{-i\theta}$$

$$|z| = \sqrt{z z^*} = r$$

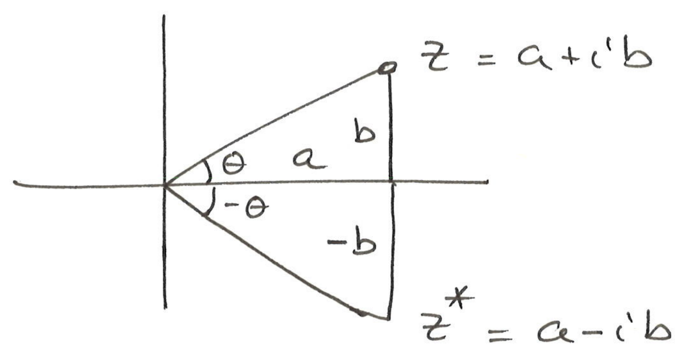
$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

Conjugation:

$$z^* = r e^{-i\theta} = a - ib$$

$$z = r e^{+i\theta} = a + ib$$



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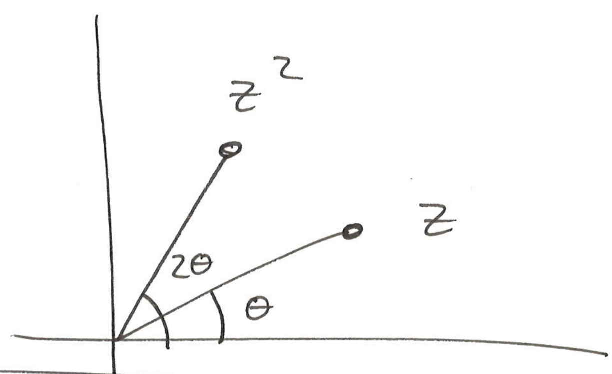
z^2 : Squared:

Consider special case: $r=1$

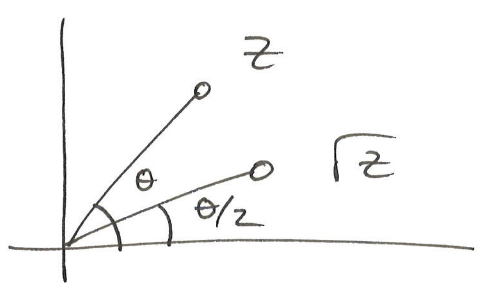
$$z = e^{i\theta} \;;\; z^2 = e^{2i\theta}$$



Square Root: $\sqrt{z}$

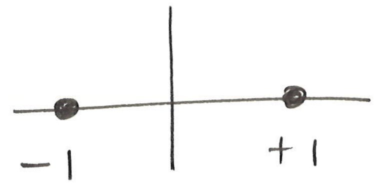
Take $r=1$

$$z = e^{i\theta} \qquad \sqrt{z} = e^{i\theta/2}$$

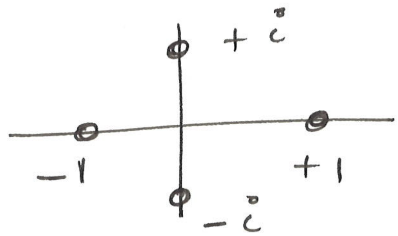


Roots of Unity

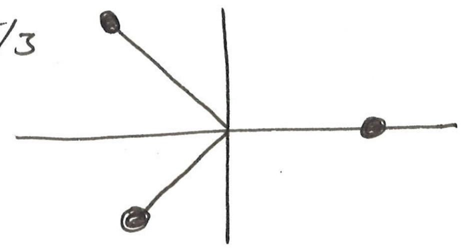
$$\sqrt{1} = \pm 1$$
$$= e^{0}, e^{i\pi}$$



$$\sqrt[4]{1} = \pm 1, \pm i$$
$$= e^{0}, e^{i\pi}, e^{\pm i\pi/2}$$



$$\sqrt[3]{1} = e^{0}, e^{i2\pi/3}, e^{-i2\pi/3}$$



In General:

$$\sqrt[n]{1} = \exp \text{ of } \quad i \frac{2\pi}{n} \cdot K$$

$$\text{where} \quad K = \{0, \ldots, n-1\}$$

Logarithm:

$$\ln z = \ln r e^{i\theta} = \ln(r) + \ln e^{i\theta}$$
$$= \ln(r) + i\theta$$

Problem: $e^{0} = 1 = e^{i2\pi n}$ for n integer

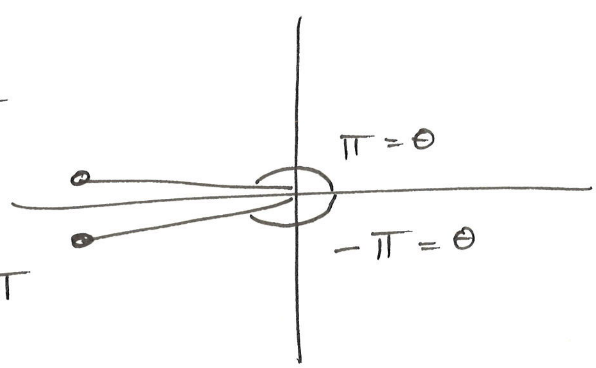
$$\Rightarrow \quad \ln e^{0} = \ln 1 = \ln e^{i2\pi n}$$
$$0 = 0 = i2\pi n$$
$$\Rightarrow \quad 0 = i2\pi = i4\pi = i6\pi = \dots$$

What to do ???

Restrict $\theta \in [\pi, -\pi)$; $\ln e^{i\theta} = i\theta$

$\ln e^{i\pi} = \pi$

$\ln e^{-i\pi} = -\pi$



Difference : 2π

The negative x axis is the $\boxed{\text{cut line}}$

$\ln z$ is not contnious across cut line!!!

Sqrt:

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2}$$

Problem $\quad 1 = e^{0} = e^{i2\pi}$

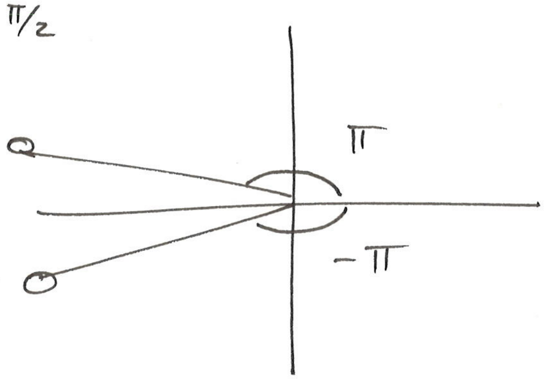
$$1 = \sqrt{1} = e^{0} = e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$$

$$\Rightarrow \quad +1 = -1$$

What to Do: $\quad$ Branch cut along negative Re axis.

$$\sqrt{z} = \sqrt{r}\, e^{+i\pi/2}$$

$$\sqrt{z} = \sqrt{r}\, e^{-i\pi/2}$$



Difference $\quad \sqrt{r}\left(e^{i\pi/2} - e^{-i\pi/2}\right)$

$$= \sqrt{r}\,(i - (-i)) = 2i\sqrt{r}$$