Group Theory Ch. 4

Definition of Group

1) Closure: $a, b \in G$, then $ab \in G$

2) Associative: $(ab)c = a(bc)$

3) Identity: $\mathbb{1} a = a \mathbb{1} = a$

4) Inverse: $a a^{-1} = a^{-1} a = \mathbb{1}$

Example: $\{\mathbb{1}, i, -1, -i\}$

1) Closure: Make multiplication table:

	1	i	-1	$-i$
1	$\mathbb{1}$	i	-1	$-i$
i	i	-1	$-i$	$\mathbb{1}$
-1	-1	$-i$	$+1$	$+i$
$-i$	$-i$	1	$+i$	-1

2) Associativity: skip

3) Identity: $1\cdot(1) = 1$, $1\cdot(i) = i$, $1\cdot(-1) = -1$, $1\cdot(-i) = -i$

4) Inverse: $1(1) = 1$ $\quad i(-i) = 1$

$-1(-1) = 1$ $\quad -i(i) = 1$

Abelian: or Commutative

$$ab = ba$$

$$ab - ba = 0$$

$$[a, b] = 0$$

Non-Abelian or Non-Commutative

$$ab \neq ba$$

$$ab - ba \neq 0$$

$$[a, b] \neq 0$$

Commutator:

$$[a,b] = ab - ba$$

Abelian Example:

$$2*3 = 3*2 = 6$$

$$[2, 3] = 0$$

$$2*3 - 3*2 = 0$$

Non-Abelian Example:

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$ab = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$ba = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\therefore \quad ab \neq ba$$

$$[a, b] \neq 0$$

Sub-Group:

$G = \{1, i, -1, -i\}$ group

$H = \{1, -1\}$ subgroup of G

Check 1) Closure →

	1	−1
1	1	−1
−1	−1	1

2) Associativity (skip)

3) Identity: $\mathbb{1}(1) = 1$, $1 \cdot (-1) = -1$

4) Inverse $\mathbb{1}(1) = 1$, $(-1)\cdot(-1) = +1$

Group H just ~~…~~ flips sign.

Supposse we don't care about the sign.

G/H (G mod H)

we don't distinguish $\{1, -1\}$ ← cosets
$\{i, -i\}$ ←

Or, mod H, $1 = -1$ and $i = -i$

Note: Order of $G = 4$
Order of $H = 2$
Order of $G/H = 4/2 = 2$

Example: The coffee group

$G_1 =$ Sugar $\{1, s\}$

$G_2 =$ Caffine $\{1, c\}$

$$G = G_1 \otimes G_2 = \{1, s, c, sc\}$$

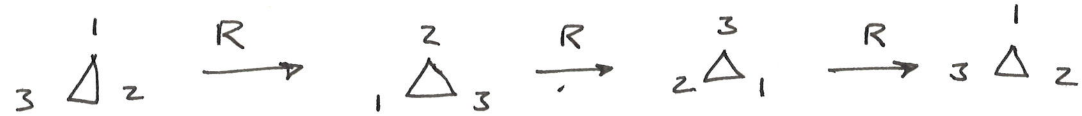
If I don't care about caffine:

$$1 = c, \quad s = sc$$

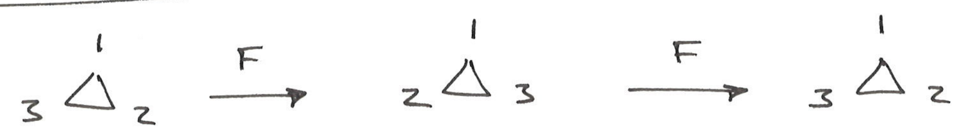
$\{1, c\}$ $\{s, sc\}$ 2 cosets

G/G_2

Dihedral Group D_3:



$G_1 = \{1, R, R^2\}$ This is a group of order 3



$G_2 = \{1, F\}$ Group of order = 2

Consider $D_3 = G_1 \otimes G_2$

$$
\begin{tikzcd}
\boxed{123} \arrow[r, "R"] \arrow[d, "F"'] & \boxed{231} \arrow[r, "R"] \arrow[d, "F"] & \boxed{312} \arrow[d, "F"] \\
\boxed{132} & \boxed{213} \arrow[l, "R"'] & \boxed{321} \arrow[l, "R"']
\end{tikzcd}
$$

Note: $RF \neq FR \Rightarrow [R, F] \neq 0$

<u>Non-Abelian</u>

Rubiks Group :

12 Edge Cubes : $12!$ Location

2^{12-1} orientation

8 Corner Cubes : $8!$ Location

3^{8-1} orientation

Total: $(12!)\,\cancel{\#}2^{12} \cdot (8!)\, 3^{8} \,//\, 12$

$12 \to 2 \cdot 3 \cdot 2$

Edge, Corner, Eveness

$= 43252003274489856000$

$\doteq 4 \times 10^{19}$

Similarity TX :

ABA^{-1} is similar to B

$ABA^{-1} \sim B$

Matrix Representation:

$$G = \{1, i, -1, -i\} \equiv \{\mathbb{1}, R, R^2, R^3\}$$

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv R^4$$

$$R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad R^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

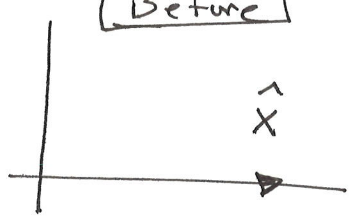
Note: R is a rotation by 90°

Rotation:

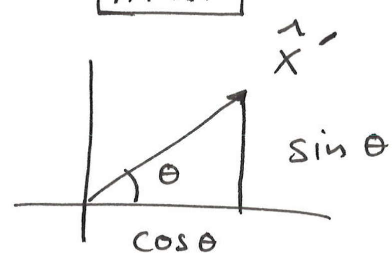
Rotate a unit vector $\hat{X}$ by a general rotation: $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$

$$\hat{X}' = R(\hat{X}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ +\sin\theta \end{pmatrix}$$

Before



After



Note: I picked the sign of $R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ opposite to make things easy.

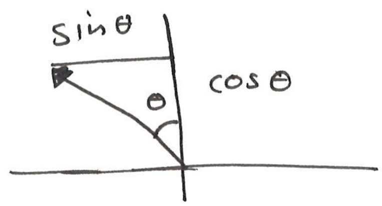
Depends on def. of active vs. passive Tx.
CF. Section 3.3, p. 185

Rotate $\hat{y}$ by R:

$$\hat{y}' = R\,\hat{y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$\boxed{\text{Before}}$

$\boxed{\text{After}}$



Note: Rotation in opposite direction $\theta \to -\theta$

$$R(-\theta) \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Exponential Form of Rotation:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{i\sigma_2\theta} = \mathbb{1}\cos\theta + i\sigma_2\sin\theta$$

σ_2 is the generator of the transformation.

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Note $\sigma_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

Prove: $e^{i\sigma_z\theta} = \mathbb{1}\cos\theta + i\sigma_z\sin\theta$

Series Expansion: $e^{A} \cong 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!}$

$$e^{i\sigma_z\theta} \cong \mathbb{1} + i\sigma_z\theta + \frac{(i\sigma_z\theta)^2}{2!} + \frac{(i\sigma_z\theta)^3}{3!}$$

$$= \mathbb{1} + i\sigma_z\theta + \left(-\frac{\theta^2}{2}\right)\mathbb{1} + \left(-\frac{i\theta^3}{6}\right)\sigma_z$$

using $\sigma_z^2 = \mathbb{1}$

$$= \mathbb{1}\underbrace{\left(1 - \frac{\theta^2}{2} + \dots\right)}_{\sim\cos\theta} + i\sigma_z\underbrace{\left(\theta - \frac{\theta^3}{6}\right)}_{\sim\sin\theta}$$

$$\equiv \mathbb{1}\cos\theta + i\sigma_z\sin\theta$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\theta + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\theta$$

$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$