1 Mellin Transform Example:

1.1 Mellin Transform Definition:

Following the Wikipedia convention:

$$\begin{aligned}
\widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, f(x) \\
f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\ x^{-n}\, \widetilde{f}(n)
\end{aligned}$$

Here, $f(x)$ is the function in x-space,and $\widetilde{f}(n)$ is the Mellin transform in n-space.

1.2 Cauchy's integral formula:

We will make use of Cauchy's integral formula:

$$g(a) = \frac{1}{2\pi i}\oint \frac{g(z)}{z-a}\, dz$$

1.3 Single term: :$f(x) = x^3$.

Let's start with just a single term:$f(x) = x^3$.

$$\begin{aligned}
\widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, x^3 \\
&= \int_0^1 dx\, x^{n+2} \\
&= \frac{1}{n+3}
\end{aligned}$$

Now let's take the inverse Mellin transform. (Note the pole/singularity is at a negative integer.)

$$\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\ x^{-n}\, \widetilde{f}(n) \\
&= \frac{1}{2\pi i}\oint \frac{g(n)}{n-(-3)}\, dn \\
&= g(-3) = x^{-(-3)} = x^3
\end{aligned}$$

where $g(n) = x^{-n}$.

1.4 General term: :$f(x) = x^a$.

For a general term:$f(x) = x^a$.

$$\begin{aligned}
\widetilde{f}(n) &= \int_0^1 dx\, x^{n-1}\, x^a \\
&= \int_0^1 dx\, x^{n+a-1} \\
&= \frac{1}{n+a}
\end{aligned}$$

Now let's take the inverse Mellin transform. (Note, we'll assume $a > 0$ so that the pole/singularity is at a negative integer.)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\; x^{-n}\,\widetilde{f}(n) \\
&= \frac{1}{2\pi i}\oint \frac{g(n)}{n-(-a)}\,dn \\
&= g(-3) = x^{-(-a)} = x^{a}
\end{aligned}
$$

where $g(n) = x^{-n}$.

1.5 General Polynomial

We can apply to a general polynomial:

$$
\begin{aligned}
f(x) &= \sum_k c_k\, x^k \\
\widetilde{f}(n) &= \sum_k \frac{c_k}{n+k}
\end{aligned}
$$

1.6 Factorize convolution

Consider the convolution below; this is a function of z, as x and y are integrated over.

$$
[f\otimes g]\,(z) = \int dx \int dy\, f(x)\, g(y)\, \delta(z - x\, y)
$$

Let's take the Mellin transform

$$
\begin{aligned}
\int z^{n-1}\,[f\otimes g]\,(z) &= \int dz\, z^{n-1}\int dx \int dy\, f(x)\, g(y)\, \delta(z - x\, y) \\
&= \left[\int dx\, x^{n-1}\, f(x)\right]\left[\int dy\, y^{n-1}\, g(y)\right]\left[\int dz\, \delta(z - x\, y)\right] \\
&= \widetilde{f}(n)\,\widetilde{g}(n)
\end{aligned}
$$

Here, we have used the delta function to replace $z^{n-1} \to x^{n-1}y^{n-1}$, and then the delta function eliminates the $\int dz$ integral. Thus, we see that in Mellin n-space, the convolution becomes a simple production of the Mellin moments.

(11.19)

$$
\begin{aligned}
&(a)\quad \delta(-x) = \delta(x) \text{ and } \delta(x-a) = \delta(a-x); \\
&(b)\quad \delta'(-x) = -\delta'(x) \text{ and } \delta'(x-a) = -\delta'(a-x); \\
&(c)\quad \delta(ax) = \frac{1}{|a|}\,\delta(x),\;\; a \neq 0; \\
&(d)\quad \delta[(x-a)(x-b)] = \frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)],\;\; a \neq b; \\
&(e)\quad \delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \quad \text{if } f(x_i) = 0 \text{ and } f'(x_i) \neq 0.
\end{aligned}
$$