Physics 7311 — Professor Scalise
Homework Assignment #5 — Due: Wednesday 2 October 2002

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13. Suppose we have an assembly of uniform balls of charge. Then $\rho(\vec{r}) = \sum_i q_i \Delta^3(\vec{r} - \vec{r_i})$ where

$$\Delta^3(\vec{r} - \vec{r_i}) = \begin{cases} \frac{1}{\frac{4}{3}\pi a^3}, & \vec{r} \text{ in a sphere of radius } a \text{ centered at } \vec{r_i} \\ 0, & \vec{r} \text{ outside the sphere defined above} \end{cases}$$

Thus (assuming that $|\vec{r_i} - \vec{r_j}| > 2a$ for $i \neq j$) the electrostatic potential energy of the system is

$$W = \frac{1}{2}\sum_{i\neq j} q_i q_j \int dV \int dV' \frac{\Delta^3(\vec{r} - \vec{r_i})\Delta^3(\vec{r}\,' - \vec{r_j})}{|\vec{r} - \vec{r}\,'|} + \frac{1}{2}\sum_i q_i^2 \int dV \int dV' \frac{\Delta^3(\vec{r} - \vec{r_i})\Delta^3(\vec{r}\,' - \vec{r_i})}{|\vec{r} - \vec{r}\,'|} \quad .$$

The last contribution is the sum of **self-energies** of the uniform balls of charge. Calculate the self-energy of the ith ball of charge – i.e. calculate

$$W_i^{\text{self}} = \frac{1}{2} q_i^2 \int dV \int dV' \frac{\Delta^3(\vec{r} - \vec{r_i})\Delta^3(\vec{r}\,' - \vec{r_i})}{|\vec{r} - \vec{r}\,'|} \quad .$$

What happens when $a \to 0$? Hint: Shift the origin of coordinates and then put W_i^{self} in the form

$$W_i^{\text{self}} = \frac{1}{2}\int dV \rho_i(\vec{r})\Phi_i^{\text{self}}(\vec{r})$$

with

$$\rho_i(\vec{r}) = q_i \Delta^3(\vec{r})$$

and

$$\Phi_i^{\text{self}}(\vec{r}) = \int dV' \frac{\rho_i(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

and calculate $\Phi_i^{\text{self}}(\vec{r})$ from Gauss' Law.

14. Prove that $C_{ij} = C_{ji}$ where

$$C_{ij} = -\oint_{S_i} \frac{dS}{4\pi} \oint_{S_j} \frac{dS'}{4\pi} \hat{n}_i \cdot \vec{\nabla}_r \;\; \hat{n}_j' \cdot \vec{\nabla}_{r'} G_D(\vec{r}, \vec{r}\,') \quad .$$

(over)

15. We often refer to the capacitance C of a pair of conductors. In particular, we put $q_1 = Q$ and $q_2 = -Q$. (equal and oppositely charges conductors). The capacitance is defined as

$$C = \frac{Q}{\Phi_1 - \Phi_2} \quad .$$

(a) Determine C in terms of C_{11}, C_{22}, and C_{12}.

(b) Suppose that conductor #1 is completely enclosed by conductor #2. Prove that $C_{12} = -C_{11}$ in this case. Find the capacitance in this case. Hint: We require $\Phi(\vec{r})$ (the potential at a point $\vec{r}$ due to both conductors) to vanish as $r \to \infty$. With Q on conductor #1 and $-Q$ on conductor #2, what is the electric field outside conductor #2? What does that say about Φ_2?

(c) Consider the case of two concentric spheres of radii R_1 and R_2 with $R_1 < R_2$ and on which there are charges q_1 and q_2, respectively. Do **NOT** assume $q_1 = -q_2$! Find explicit expressions for C_{11}, C_{22}, and C_{12} using elementary physics (e.g. Gauss' Law) together with definitions of the C_{ij} ($q_i = \sum_j C_{ij}\Phi_j$). Comment on your results in light of part (b).

(d) What simplification takes place in part (a) if the two conductors are identical in shape and size?