3) $C = +k^2$ $\quad k$ real, positive

$$Z''_{(z)} + k^2 Z_{(z)} = 0 \Rightarrow Z_{(z)} = a''_n \cos(kz) + b''_n \sin(kz)$$

$$\stackrel{\text{or}}{\equiv} c_n e^{in\varphi} \quad \text{for complex } c_n$$

$$\frac{1}{\rho}(\rho R')' - \left(k^2 + \frac{n^2}{\rho^2}\right)R = 0 \Rightarrow R_{(\rho)} = d_1'' I_n(k\rho) + d_2'' K_n(k\rho)$$

where

$I_n(u)$ is the <u>modified</u> Bessel function of the first kind and

$K_n(u)$ is the <u>modified</u> Bessel function of the second kind. $I_n$ and $K_n$ are <u>not</u> complete.

The integer $n$ is the <u>order</u> of the Bessel function. The modified Bessel functions are defined for negative order by

$$I_{-n}(u) \equiv (-1)^n I_n(u)$$

$$K_{-n}(u) \equiv (-1)^n K_n(u)$$

$I_n(u)$ are <u>growth</u> functions and $K_n(u)$ are <u>decay</u> functions. They are analogous to exponentials — in fact;

$$I_n(u) \xrightarrow[u\to\infty]{} \frac{1}{\sqrt{2\pi u}}\, e^{u} \qquad n \geq 0$$

$$K_n(u) \xrightarrow[u\to\infty]{} \frac{1}{\sqrt{2\pi u}}\, e^{-u} \qquad n \geq 0$$

---

$I_n(u)$ is well-behaved at the origin

$$I_n(u) \xrightarrow[u\to 0]{} \frac{u^n}{2^n} \qquad n \geq 0$$

while $K_n(u)$ is divergent there

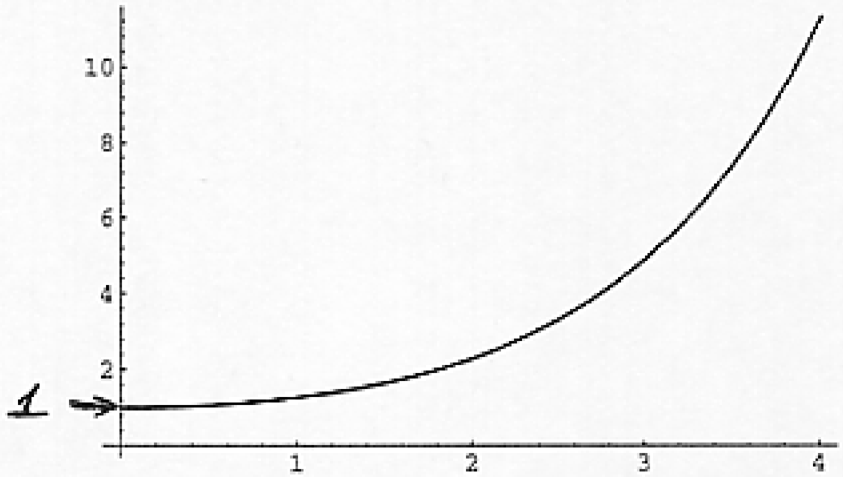
$$K_0(u) \xrightarrow[u\to 0]{} \ln\left(\frac{2}{u}\right) - \gamma_E$$

$$K_n(u) \xrightarrow[u\to 0]{} \frac{1}{2}\,\frac{1}{(n-1)!}\left(\frac{2}{u}\right)^n \qquad n \geq 1$$

Hence, when the $g=0$ axis is included in the region of interest $V$, we must exclude the $K_n(u)$.

*In[26]:=*

```
Plot[BesselI[0,u],{u,0,4}]
```




*Out[26]=*

```
-Graphics-
```

*In[22]:=*

```
BesselI[0,0]
```

*Out[22]=*

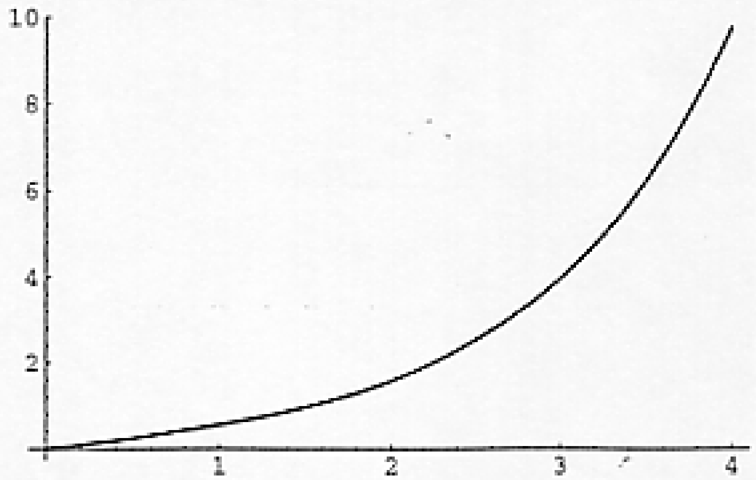
```
1
```

$I_0(0) = 1$ while all the others orders satisfy

$I_n(0) = 0.$

*In[27]:=*

```
Plot[BesselI[1,u],{u,0,4}]
```



*Out[27]=*

```
-Graphics-
```

*In[28]:=*

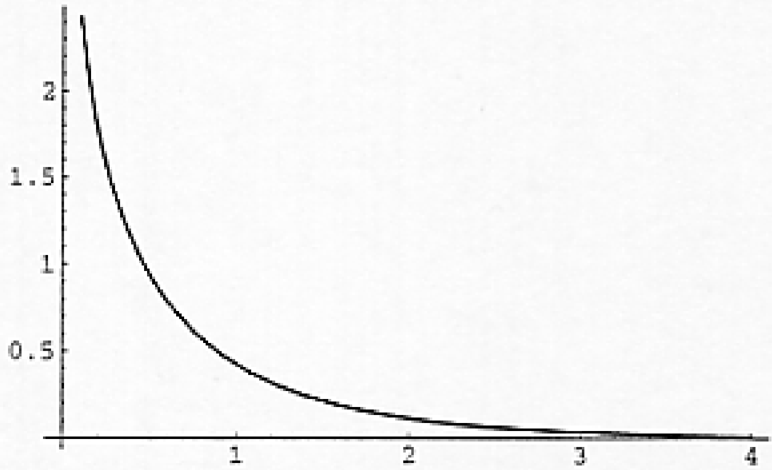
```
BesselI[1,0]
```

*Out[28]=*

```
0
```

*In[30]:=*

```
Plot[BesselK[0,u],{u,0.1,4}]
```




*Out[30]=*

```
-Graphics-
```

*In[31]:=*

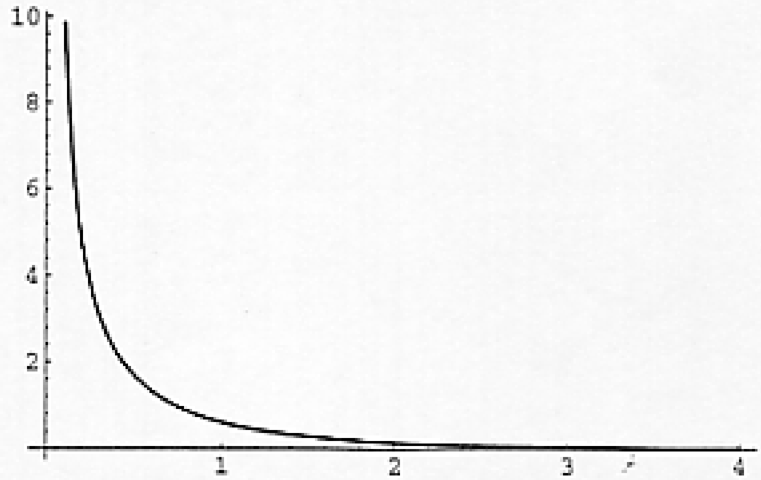
```
BesselK[0,0]
```

*Out[31]=*

```
Infinity
```

*In[33]:=*

```
Plot[BesselK[1,u],{u,0.1,4}]
```




*Out[33]=*

```
-Graphics-
```

*In[34]:=*

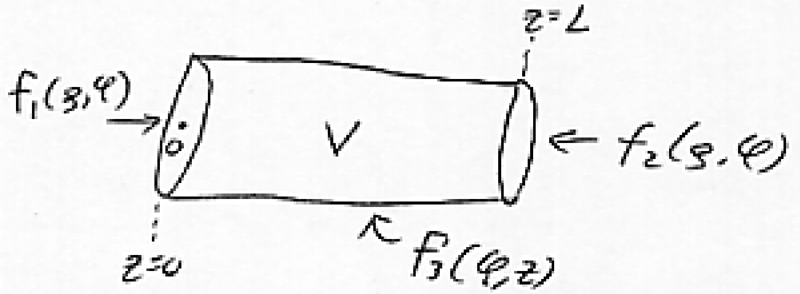
```
BesselK[1,0]
```

*Out[34]=*

```
ComplexInfinity
```

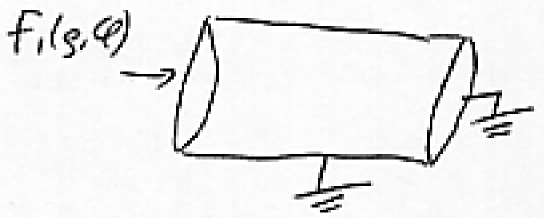
Consider the following example problem:

Find the potential inside a full cylinder of length $L$ and radius $a$, given the following Dirichlet boundary conditions:




The solution is $\Phi(\rho,\varphi,z) = \Phi_1(\rho,\varphi,z) + \Phi_2(\rho,\varphi,z) + \Phi_3(\rho,\varphi,z)$

where the partial solution $\Phi_1(\rho,\varphi,z)$ is the solution to the following problem:

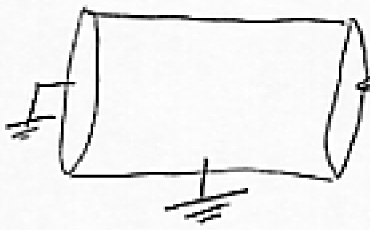



We can satisfy Laplace's equation $\nabla^2\Phi_1(\rho,\varphi,z)=0$ and the boundary conditions with the series:

$$\Phi_1(\rho,\varphi,z) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{\infty} J_n\left(u_{ns}\frac{\rho}{a}\right)\left[A_{ns}\cos(n\varphi) + B_{ns}\sin(n\varphi)\right]\sinh\left(u_{ns}\frac{L-z}{a}\right)$$

We always have oscillating solutions in $\varphi$, but now we also want oscillating functions of $\rho$ — thus we are forced to have exponential functions of $z$. $\boxed{k = u_{ns}\, a}$

The partial solution $\Phi_2(\rho,\varphi,z)$ is the potential inside the cylinder with the following Dirichlet boundary conditions:

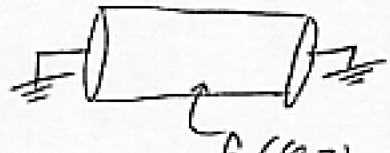



The series solution is:

$$\Phi_2(\rho,\varphi,z) = \sum_{n=-\infty}^{\infty}\sum_{s=1}^{\infty} J_n\left(u_{ns}\frac{\rho}{a}\right)\left[A'_{ns}\cos(n\varphi) + B'_{ns}\sin(n\varphi)\right]\sinh\left(u_{ns}\frac{z}{a}\right)$$

---

The partial solution $\Phi_3(\rho,\varphi,z)$ satisfies Laplace's equation are the following boundary conditions:




This time, we want a complete set of functions of $z$. The functions of $\rho$ are growth and decay.

$$\Phi_3(\rho,\varphi,z) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} I_n\left(\frac{m\pi\rho}{L}\right)\left[A''_{nm}\cos(n\varphi) + B''_{nm}\sin(n\varphi)\right]\sin\left(\frac{m\pi z}{L}\right)$$

$$\boxed{k = \frac{m\pi}{L}}$$

---

If we were considering instead the region <u>between</u> two finite cylinders we would include $N_n(u)$ and $K_n(u)$ in the series solution.

The coefficients $A_{ns}, B_{ns}; A'_{ns}, B'_{ns}; A''_{nm}$ and $B''_{nm}$ are determined from Fourier analysis on the bounding surfaces. This will be a homework problem.

---

Dirichlet Green function for a cylinder

Consider the set of 3-dimensional orthonormal functions:

$$\psi_{nms}(\rho,\varphi,z) = \sqrt{\frac{2}{\pi L}}\,\frac{1}{a\,J_{n+1}(x_{ns})}\,e^{in\varphi}\,J_n\!\left(x_{ns}\frac{\rho}{a}\right)\sin\!\left(\frac{m\pi z}{L}\right)$$

this is not a solution to Laplace's equation.

We have written $e^{in\varphi}$ as a short hand for the sines and cosines: $e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi)$.

$\psi_{nms}(\rho,\varphi,z)$ is designed to vanish at all the bounding surfaces.

Orthonormality

$$\int_0^a \rho\, d\rho \int_0^{2\pi} d\varphi \int_0^L dz\; \psi^*_{n'm's'}(\rho,\varphi,z)\,\psi_{nms}(\rho,\varphi,z) = \delta_{nn'}\delta_{mm'}\delta_{ss'}$$

note

$n = 0, \pm 1, \pm 2, \ldots$
$m = 1, 2, 3, \ldots$
$s = 1, 2, 3, \ldots$

This set is complete with respect to all functions periodic in $\varphi$ (period $2\pi$) and $z$ (period $2L$).

There is a closure relation derived by analogy with the Cartesian case.

$$\sum_{nms} \psi^*_{nms}(\rho', \varphi', z')\, \psi_{nms}(\rho, \varphi, z) = \delta^3(\vec{r} - \vec{r}\,')$$

where in cylindrical coordinates

$$\delta^3(\vec{r} - \vec{r}\,') = \frac{1}{\rho}\, \delta(\rho - \rho')\, \delta(\varphi - \varphi')\, \delta(z - z')$$

---

We can expand the Dirichlet Green function as

$$G_D(\vec{r}, \vec{r}\,') = \sum_{nms} \psi^*_{nms}(\rho', \varphi', z')\, \psi_{nms}(\rho, \varphi, z)\, G_{nms}$$

this satisfies

1) symmetry: $G_D(\vec{r}, \vec{r}\,') = G_D(\vec{r}\,', \vec{r})$

2) $G_D(\vec{r}, \vec{r}\,') = 0$ for $\vec{r}, \vec{r}\,'$ on $S$

the last requirement

3) $\nabla^2 G_D(\vec{r}, \vec{r}\,') = -4\pi\, \delta^3(\vec{r} - \vec{r}\,')$

determines the constants $G_{nms}$.

$$G_{nms} = \frac{4\pi}{\left(\frac{x_{ns}}{a}\right)^2 + \left(\frac{m\pi}{L}\right)^2}$$

RESERVE

— End #13 —