7 November 95

The vector potential that we introduced last time is <u>minimal</u> in the following sense. Since we will take the curl of the vector potential in order to find the magnetic field, we are free to add to $\vec{A}(\vec{r})$ the gradient of any scalar function.

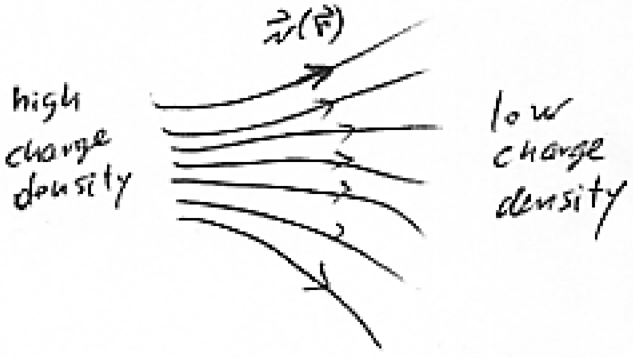
$$\vec{A}_0(\vec{r}) = \frac{1}{c}\sum_{i=1}^{N} \frac{q_i \vec{v}_i}{|\vec{r}-\vec{r}_i|} \qquad \underline{\text{minimal } \vec{A}(\vec{r})}$$

$$\vec{A}(\vec{r}) = \vec{A}_0(\vec{r}) + \vec{\nabla}\psi(\vec{r})$$

These two vector potentials give rise to the same $\vec{B}$ field. More about this later.

We now define the continuum version of the minimal vector potential:

We consider a charged "fluid" and since we want to treat magneto<u>statics</u> we will not allow the charge density $\rho(\vec{r})$ to change in time and the velocity field will also be constant. Imagine the charge flowing in many tiny wires (to keep the charges on track and moving with the specified velocity).



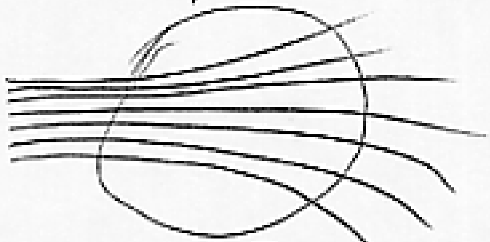
$$\vec{A}_0(\vec{r}) = \frac{1}{c}\int dV' \frac{\rho(\vec{r}')\,\vec{v}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The combination $\rho(\vec{r}')\vec{v}(\vec{r}')$ is defined to be $\vec{J}(\vec{r}')$ the "current density." This is the current per unit <u>area</u> not volume.

In the steady state:

$\oint_S dS\, \hat{n}\cdot\vec{J}(\vec{r})$ is the rate at which charge leaves the volume V bounded by S.

$$= -\frac{\partial Q}{\partial t}$$



As many wires leave S as enter S. Otherwise:

a) charge would be building up in S against our assumption that $\rho(\vec{r})$ is <u>constant</u>,

<u>or</u>

b) there would be a <u>source</u> or <u>sink</u> of charge in S, which would violate charge conservation. Charge is neither created nor destroyed in V.

We can write the closed surface integral as a volume integral using the divergence theorem

$$\oint_S dS\, \hat{n}\cdot\vec{J}(\vec{r}) = \int_V dV\, \vec{\nabla}\cdot\vec{J}(\vec{r}) = -\frac{\partial Q}{\partial t} = -\int_V dV\, \frac{\partial \rho(\vec{r})}{\partial t}$$

Combining these we get

$$\int_V dV \left[\vec{\nabla}\cdot\vec{J}(\vec{r}) + \frac{\partial \rho(\vec{r})}{\partial t} \right] = 0$$

This statement must be true for *any* volume V. The only way this can happen is if the integrand itself vanishes:

$$\boxed{\vec{\nabla}\cdot\vec{J}(\vec{r}) + \frac{\partial\rho(\vec{r})}{\partial t} = 0}$$

This is the continuity equation.

Since we are limiting the discussion to statics and $\frac{\partial\rho(\vec{r})}{\partial t} = 0$ we have in this special case:

$$\vec{\nabla}\cdot\vec{J}(\vec{r}) = 0$$

We will use this result shortly.

Take the divergence of the minimal vector potential:

$$\vec{\nabla}\cdot\vec{A}_0(\vec{r}) = \int dV'\,\frac{\vec{J}(\vec{r}')}{c}\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{c}\int dV'\,\vec{J}(\vec{r}')\cdot\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r}'|}$$

change the gradient with respect to $\vec{r}$ to the gradient with respect to $\vec{r}'$ and change sign.

Integrate by parts

$$= -\frac{1}{c}\int dV'\,\vec{\nabla}_{r'}\cdot\left[\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] + \frac{1}{c}\int dV'\,\frac{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The first integral contains a total derivative and can be written as a surface integral. We choose the surface large enough so that the current density $\vec{J}(\vec{r}')$ on S vanishes. In the second term, $\vec{\nabla}\cdot\vec{J}(\vec{r}) = 0$.

The result is that the divergence of the minimal vector potential vanishes:

$$\vec{\nabla}\cdot\vec{A}_0(\vec{r}) = 0$$

We now derive two Maxwell Equations in vacuum:

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}_0(\vec{r}) \quad \text{so} \quad \vec{\nabla}\cdot\vec{B}(\vec{r}) = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}_0(\vec{r})\right) = 0$$

since the divergence of the curl of anything is zero.

$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0}$$

This is called the magnetic Gauss' Law,

and expresses the fact that magnetic charge does not exist. We see this by considering the electric version of Gauss' Law.

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi\rho_e(\vec{r})$$ electric charge density

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Now consider the curl of the magnetic field.

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \vec{\nabla}\times\left[\vec{\nabla}\times\vec{A}_0(\vec{r})\right] = \vec{\nabla}\left[\cancelto{0}{\vec{\nabla}\cdot\vec{A}_0(\vec{r})}\right] - \nabla^2\vec{A}_0(\vec{r})$$

$$= -\nabla^2\vec{A}_0(\vec{r})$$

$$-\nabla^2\vec{A}_0(\vec{r}) = -\frac{1}{c}\int dV'\,\vec{J}(\vec{r}\,')\,\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|} = +\frac{4\pi}{c}\int dV'\,\vec{J}(\vec{r}\,')\,\delta^3(\vec{r}-\vec{r}\,')$$

$$= \frac{4\pi}{c}\vec{J}(\vec{r})$$

So we have:

$$\boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{4\pi}{c}\,\vec{J}_e(\vec{r})}$$

This is Ampere's Law

electric current density

Recall the analogous Maxwell Equation from electro-statics

$$\vec{\nabla}\times\vec{E}(\vec{r}) = 0 = 4\pi\,\cancel{\vec{J}_m(\vec{r})}^{\,0}$$

This is Faraday's Law

there is no magnetic current since there is no magnetic charge.

The *integral* form of Ampere's law is

$$\int_S dS\,\hat{n}\cdot\vec{\nabla}\times\vec{B}(\vec{r}) = \int_S dS\,\hat{n}\cdot\vec{J}(\vec{r})\,\frac{4\pi}{c}$$

where S is an *open* surface. The following analogy may be useful.

An open *volume* is bounded by a closed *surface*.

An open *surface* is bounded by a closed *curve*.

An open *curve* is bounded by a two points.

We use Stoke's Theorem to write

$$\int_S dS\ \hat{n}\cdot\vec{\nabla}\times\vec{B}(\vec{r}) = \oint_C d\vec{l}\cdot\vec{B}(\vec{r})$$

The right-hand-side of Ampere's law is

$$\int_S dS\ \hat{n}\cdot\vec{J}(\vec{r})\,\frac{4\pi}{c} = \frac{4\pi}{c}\, I_{\text{enclosed by the curve } C}$$

↖ current passing through surface S.

$$\boxed{\oint_C d\vec{l}\cdot\vec{B}(\vec{r}) = \frac{4\pi}{c} I_{enc}}$$

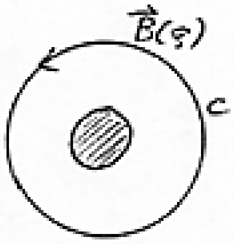
$\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$ ⟹ lines of magnetic field do not diverge from or converge to points;

$\vec{\nabla}\times\vec{B}(\vec{r}) \neq 0$ ⟶ lines of magnetic field close on themselves.

Ampere's Law and symmetry can be used just as we use Gauss' Law and symmetry in electrostatics.

We solve a simple example problem:

Consider a long straight wire of radius $\underline{a}$ carrying a uniform current density $\vec{J}$. As the curve C in Ampere's law, choose a circle concentric with the wire at radius ρ.

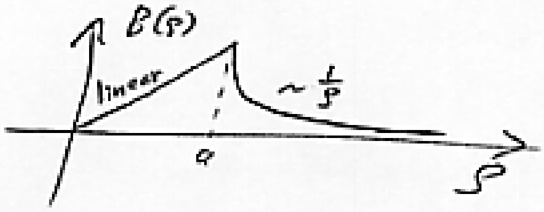


By symmetry, the lines of magnetic field are circles concentric with the wire, and the magnitude of B can depend only on ρ, not z or φ.

$$\oint_C d\vec{l} \cdot \vec{B}(\vec{r}) = 2\pi\rho\, B(\rho) = \frac{4\pi}{c} I_{enc} = \begin{cases} \frac{4\pi}{c} I \frac{\rho^2}{a^2} & , \rho < a \\ \frac{4\pi}{c} I & , \rho > a \end{cases}$$

$I = J\pi a^2$ = total current

$$\Rightarrow B(\rho) = \begin{cases} \frac{2I\rho}{c\,a^2} & , \rho < a \\ \frac{2I}{c\rho} & , \rho > a \end{cases}$$



Axially symmetric current distributions in the wire are also easily handled.

Gauge Invariance

A vector potential which differs from the minimal vector potential gives rise to the same $\vec{B}$ field

$$\vec{A}(\vec{r}) = \vec{A}_0(\vec{r}) + \vec{\nabla}\psi(\vec{r}) \qquad \text{for any } \psi(\vec{r})$$

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \vec{\nabla}\times\vec{A}_0(\vec{r}) \qquad \text{since} \quad \vec{\nabla}\times\vec{\nabla}\psi = 0$$

Since the vector potential is not measurable (not physical), the gauge function $\psi(\vec{r})$ cannot be determined by the problem. We can use this arbitrariness to "choose a gauge" with affecting the magnetic field.

The choice $\vec{\nabla}\cdot\vec{A}(\vec{r}) = 0$ is called Coulomb gauge.

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = 0 = \cancel{\vec{\nabla}\cdot\vec{A}_0(\vec{r})}^{0} + \nabla^2\psi(\vec{r}) \implies \nabla^2\psi(\vec{r}) = 0$$

So we see that even within Coulomb gauge, there is still some arbitrariness. We can add to $\vec{A}_0(\vec{r})$ the gradient of any solution to Laplace's equation

$$\vec{A}(\vec{r}) = \vec{A}_0(\vec{r}) + \vec{\nabla}\psi(\vec{r}) \qquad \text{where } \nabla^2\psi(\vec{r}) = 0$$

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \vec{\nabla}\cdot\vec{A}_0(\vec{r}) = 0$$

The gauge condition can be used to simplify the problem - only two of the components of the vector $\vec{A}$ are independent.

Gauge invariance has become a guiding principle in the construction of Quantum Field Theories like the SU(2) Weak interaction and the SU(3) Strong Interaction (QCD).

Uniqueness Theorem for the Vector Potential

We choose Coulomb gauge at the beginning:

$$\nabla^2 \vec{A}_0(\vec{r}) = -\frac{4\pi}{c}\vec{J}(\vec{r}) \qquad \vec{\nabla}\cdot\vec{A}_0(\vec{r}) = 0$$

Now we assume <u>two</u> solutions $\vec{A}_1(\vec{r})$ and $\vec{A}_2(\vec{r})$ and as usual in these proofs we form the difference

$$\vec{\chi}(\vec{r}) = \vec{A}_1(\vec{r}) - \vec{A}_2(\vec{r})$$

then

$$\nabla^2 \vec{\chi}(\vec{r}) = 0 \qquad \text{and} \qquad \vec{\nabla}\cdot\vec{\chi}(\vec{r}) = 0$$

Next, we use an identity analagous to the Green identities that we used to prove uniqueness of the electrostatic potential $\Phi(\vec{r})$.

$$\int_V dV\, \vec{\nabla}\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right] = \oint_S dS\, \hat{n}\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right]$$

by the divergence theorem

We concentrate first on the left hand side and use

$\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$ to write

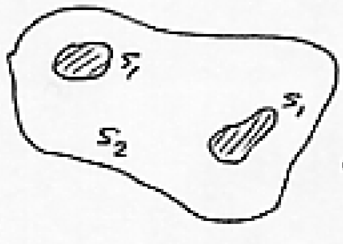
$$\vec{\nabla}\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right] = |\vec{\nabla}\times\vec{X}|^2 - \vec{X}\cdot\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{X})\right]$$

but $\vec{\nabla}\times(\vec{\nabla}\times\vec{X}) = \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{X})} - \cancel{\nabla^2\vec{X}} = 0$

$$\vec{\nabla}\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right] = |\vec{\nabla}\times\vec{X}|^2 = (\vec{\nabla}\times\vec{X})\cdot(\vec{\nabla}\times\vec{X})$$

On the right hand side, we split the closed surface S into two (possibly not simply-connected) surfaces S_1 and S_2: $S = S_1 \oplus S_2$

simply connected



not simply connected

On S_1, we use the identity: $\hat{n}_1\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right] = (\hat{n}_1\times\vec{X})\cdot\vec{\nabla}\times\vec{X}$

On S_2, we use the identity: $\hat{n}_2\cdot\left[\vec{X}\times(\vec{\nabla}\times\vec{X})\right] = -\vec{X}\cdot\left[\hat{n}_2\times(\vec{\nabla}\times\vec{X})\right]$

So we have:

$$\int_V dV\, |\vec{\nabla}\times\vec{X}|^2 = \int_{S_1} dS_1\, (\hat{n}_1\times\vec{X})\cdot\vec{\nabla}\times\vec{X} - \int_{S_2} dS_2\, \vec{X}\cdot\left[\hat{n}_2\times(\vec{\nabla}\times\vec{X})\right]$$

Now we discuss boundary conditions.

On S_1, we specify $\hat{n}_1 \times \vec{A}$ (Analog of Dirichlet boundary condition)

Both solutions $\vec{A}_1$ and $\vec{A}_2$ must satisfy this condition:

$$\hat{n}_1 \times \vec{A}_1(\vec{r}) = \hat{n}_1 \times \vec{A}_2(\vec{r}) \text{ on } S_1 \Rightarrow \hat{n}_1 \times \vec{\chi}(\vec{r}) = 0$$

therefore: $$\int_{S_1} dS_1 \, \cancelto{0}{(\hat{n}_1 \times \vec{\chi})} \cdot \vec{\nabla} \times \vec{\chi} = 0$$

On S_2, we specify $\hat{n}_2 \times \vec{B} = \hat{n}_2 \times (\vec{\nabla} \times \vec{A})$

(This is the analog of Neumann boundary conditions)

$\vec{A}_1$ and $\vec{A}_2$ both satisfy the above:

$$\hat{n}_2 \times (\vec{\nabla} \times \vec{A}_1) = \hat{n}_2 \times (\vec{\nabla} \times \vec{A}_2) \Rightarrow \hat{n}_2 \times (\vec{\nabla} \times \vec{\chi}) = 0 \text{ on } S_2$$

therefore: $$\int_{S_2} dS_2 \, \vec{\chi} \cdot \cancelto{0}{[\hat{n}_2 \times (\vec{\nabla} \times \vec{\chi})]} = 0$$

Keep in mind that the entire surface may be S_1. We can specify magnetic Dirichlet boundary conditions on the whole surface. Or, we can specify magnetic Neumann boundary conditions on the whole surface.

The final result is

$$\int_V d\Omega \, |\vec{\nabla}\times\vec{\chi}|^2 = 0 \quad \text{but } |\vec{\nabla}\times\vec{\chi}|^2 \text{ is positive-semidefinite}$$

The integral vanishes only if the integrand vanishes.

$$\Rightarrow \quad \vec{\nabla}\times\vec{\chi} = 0 \qquad \text{inside } V$$

Hence $\vec{\nabla}\times\vec{A}_1(\vec{r}) = \vec{\nabla}\times\vec{A}_2(\vec{r})$

or $\vec{A}_1(\vec{r}) = \vec{A}_2(\vec{r}) + \vec{\nabla}\psi(\vec{r})$

$\vec{A}_1$ and $\vec{A}_2$, the two solutions, are unique up to gauge invariance.

Even the electrostatic potential $\Phi(\vec{r})$ has some arbitrariness:

$$\Phi'(\vec{r}) = \Phi(\vec{r}) + \text{constant}$$

both Φ' and Φ give rise to the same electric field.

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi'(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$$

The choice $\Phi(\vec{r}) \to 0$ as $r\to\infty$ is a "gauge" choice, similar to Coulomb gauge for the vector potential $\vec{\nabla}\cdot\vec{A}(\vec{r}) = 0$.

End Lecture # 19