16. (a) Show explicitly that the expression

$$\Phi(x,y,z) = q \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{\{(x-x')^2 + (y-y')^2 + [z-(-1)^n z' + nd]^2\}^{\frac{1}{2}}}$$

for the potential between parallel grounded conducting plates with a point charge between, obtained in class by the method of images, has the property $\Phi(x,y,\frac{d}{2}) = 0 = \Phi(x,y,-\frac{d}{2})$. (Hint: Break the sum into even-n and odd-n sums.)

(b) Obtain an expression for the charge density on each of the conducting planes. At what value of (x,y) is the charge density a maximum? When $z' = 0$ (charge q equidistant from the two planes), obtain an approximate value for this maximum surface charge density. (You will need to perform a sum: by hand, or calculator, or computer – your choice!) Compare this result with the maximum charge density for a point charge q placed a distance $\frac{d}{2}$ from a **single** grounded conducting plane.

(c) **Attempt** to obtain the total charge induced on each of the conducting planes for a general location of the point charge z' between the planes by integrating the infinite series obtained in the first part of (b) for the two charge densities term by term. Comment on the result.

17. From the expression

$$\vec{F}_{\text{sphere on q}} = -\frac{q^2 a \vec{r}\,'}{(r'^2 - a^2)^2}$$

for the force on a point charge q at position $\vec{r}\,'$ outside a grounded conducting sphere of radius a, compute directly the work done by an external agent in bringing the point charge q in from infinity without acceleration. Compare this expression for the system energy with that obtained from the general expression

$$U = \frac{1}{2}\int dV\ \rho(\vec{r})\Phi(\vec{r}) \ \ .$$

In the present case, what is $\rho(\vec{r})$ and what is $\Phi(\vec{r})$? Recall that you do not want to include the infinite self-energy of the point charge.

18. The potential due to a point charge q at position $\vec{r}\,'$ outside a conducting sphere of radius a held at potential Φ_0 was found in class by the method of images to be

$$\Phi(\vec{r}) = \Phi_0 \frac{a}{r} + \frac{q}{|\vec{r} - \vec{r}\,'|} - \frac{a}{r'} \frac{q}{|\vec{r} - \frac{a^2}{r'^2}\vec{r}\,'|}$$

Find this potential by using the Green function method. [Hint: Use polar coordinates to do the surface integral in the Green function formula for the potential with $\vec{r}\,'$ (the source location) on the polar axis.]

19. Suppose that the problem of determining the potential outside a closed conductor of arbitrary shape is solved in terms of an image charge density $\rho_{\text{image}}(\vec{r})$, which is non-zero only in the interior region of the conductor. Prove that the net charge induced on the outer surface of the conductor is

$$Q = \int dV \; \rho_{\text{image}}(\vec{r}) \ .$$