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Until now, we have dealt with a collection of point monopoles. We could also consider point multipoles, the most important example of which is a distribution of

Point Dipoles

one point dipole at the origin $\qquad \Phi(\vec{r}) = \dfrac{\vec{\mu}\cdot\vec{r}}{r^3}$

one point dipole at position $\vec{r}'$ $\qquad \Phi(\vec{r}) = \dfrac{\vec{\mu}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

a collection of point dipoles $\vec{\mu}_i$ located at points $\vec{r}_i$ $\qquad \Phi(\vec{r}) = \displaystyle\sum_{i=1}^{N} \frac{\vec{\mu}_i\cdot(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$

It's a short step from the sum over a discrete dipole distribution to the integral over a continuous volume distribution of dipoles.

Let $\vec{\mathbb{P}}(\vec{r}')$ be the volume dipole moment density

Then $$\Phi(\vec{r}) = \int_V dV' \, \frac{\vec{\mathbb{P}}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

This will come up again later when we study dielectric media.

For a continuous <u>surface</u> distribution of dipoles

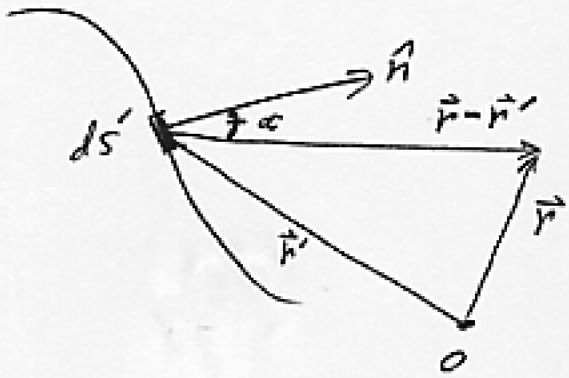
$$\Phi(\vec{r}) = \int_S dS' \frac{\vec{D}(\vec{r}') \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Let's consider a special case in which all of the dipoles are aligned <u>normal</u> to the surface S. Then

$$dS' \vec{D}(\vec{r}') = dS' D(\vec{r}') \hat{n}$$

where $\hat{n}$ is a unit vector normal to the surface.

$$\Phi(\vec{r}) = \int_S dS' \frac{D(\vec{r}') \hat{n} \cdot (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$



$$\hat{n} \cdot (\vec{r}-\vec{r}') = |\vec{r}-\vec{r}'| \cos\alpha$$

$$\Phi(\vec{r}) = \int_S dS' \frac{\cos\alpha}{|\vec{r}-\vec{r}'|^2} D(\vec{r}')$$

RESERVE

The definition of infinitesimal solid angle is

$$d\Omega' \equiv \frac{dS' \cos\alpha}{|\vec{r}-\vec{r}'|^2}$$

This is the solid angle subtended by a piece of surface area dS' located at $\vec{r}'$ as seen by an observer at $\vec{r}$.

$$\Phi(\vec{r}) = \int d\Omega' \, D(\vec{r}')$$

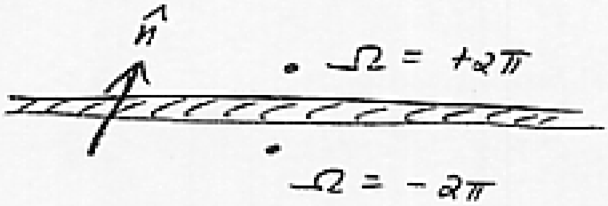
Now let's specify the problem to the case where $D(\vec{r}')$ is constant over the entire surface:

$$\Phi(\vec{r}) = D \int d\Omega' = D\Omega$$

This says that the potential is equal to the constant surface dipole moment desity D times the solid angle subtended by the whole surface, regardless of its shape!

The change in potential across a surface dipole moment density is

$$\Delta \Phi = 4\pi D$$



Very close to the surface, the solid angle subtended by the surface is half of all space, or 2π steradians. Below the surface, on the "tail" side of $\hat{n}$, the solid angle is -2π steradians because of the way we have defined $d\Omega$ in terms of the angle between $\hat{n}$ and $(\vec{r}-\vec{r}')$.

So Φ is <u>discontinuous</u> across a surface distribution of dipole moment.

We will see shortly that this is analogous the the discontinuity in $\vec{E}$ across a surface charge density.

(another word for charge is "monopole")

B) Differential and Integral Theorems of Electrostatics.

So far, we know that

$$\Phi(\vec{r}) = \int_{\substack{\text{All}\\\text{Space}}} dV' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad \text{and} \quad \vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$$

A vector calculus identity assures us that

$$\vec{\nabla}\times(\vec{\nabla} f) = 0 \qquad \forall \; f(\vec{r})$$

that is, for any scalar function f,

but $\vec{E}(\vec{r})$ is the gradient of a scalar function $\Phi(\vec{r})$

so

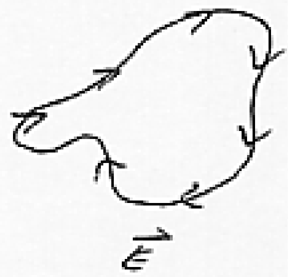
$$\boxed{\vec{\nabla}\times\vec{E} = 0}$$

Remember: this is true only for electrostatics.

What does this equation mean physically?

Electrostatic field lines never close on themselves.

<u>Proof</u>: Suppose a field line did close



Apply Stoke's Theorem

$$\oint_{\substack{\text{closed}\\ \text{field}\\ \text{line}}} d\vec{l}\cdot\vec{E} = \int_S dS\ \hat{n}\cdot\cancelto{0}{(\vec{\nabla}\times\vec{E})}$$

$$= 0$$

S is any surface (<u>open</u> surface) with the field line as its boundary.

but <u>on</u> the field line $d\vec{l}\cdot\vec{E} = dl\, E$

since $d\vec{l}$ and $\vec{E}$ point in the same direction,

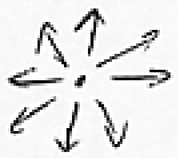
$$\oint_{\substack{\text{field}\\ \text{line}}} E\, dl = 0$$

Since the integrand is positive semi-definite, the integral can only vanish if $E \equiv 0$ everywhere along the field line.

This is certainly <u>not</u> true in general.

$\therefore$ Electrostatic field lines do not close.

Electrostatic field lines can diverge from and converge to points



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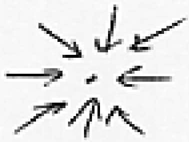


sink

Now we will discuss the divergence of $\vec{E}$, but first, we need some mathematical results:

Note: $\vec{\nabla}_r \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = - \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ ← from homework

$$\vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = \vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}'-\vec{r}|} \right) = - \frac{(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3}$$

So we have

$$\vec{\nabla}_r \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = - \vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

We need one more result, namely

$$\vec{\nabla}_r \cdot \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right) = \vec{\nabla}_{r'} \cdot \left(\frac{\vec{r}'-\vec{r}}{|\vec{r}'-\vec{r}|^3} \right) = - \vec{\nabla}_{r'} \cdot \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right)$$

Now we're ready to tackle the field $\vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = -\vec{\nabla}_r \, \Phi(\vec{r}) = -\vec{\nabla}_r \int_V dV' \rho(\vec{r}\,') \frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= -\int_V dV' \rho(\vec{r}\,') \, \vec{\nabla}_r \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

change from gradient with respect to unprimed variables to primed

$$= +\int_V dV' \rho(\vec{r}\,') \, \vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

Now take the divergence of both sides

$$\vec{\nabla}_r \cdot \vec{E}(\vec{r}) = +\int_V dV' \rho(\vec{r}\,') \, \vec{\nabla}_r \cdot \left[\vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

change to primed divergence

$$= -\int_V dV' \rho(\vec{r}\,') \, \vec{\nabla}_{r'} \cdot \left[\vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]$$

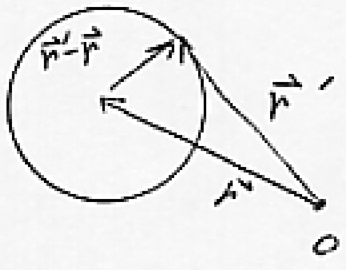
$$= -\int_V dV' \rho(\vec{r}\,') \, \nabla_{r'}^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

$\nabla_{r'}^2$ is the <u>Laplacian</u>. In Cartesian coordinates it is

$$\nabla_{r'}^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$

$\nabla^2_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right)$ is a very peculiar function:

To see this, integrate over a sphere of radius R centered at $\vec r$



$$\int_V dV'\,\nabla^2_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right)$$

$$= \int_V dV'\,\vec\nabla_{r'}\cdot\left[\vec\nabla_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right)\right]$$

use the divergence theorem

$$= \oint_{\substack{\text{sphere}\\ S^2}} dS'\,\hat n\cdot\left[\vec\nabla_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right)\right]$$

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on the surface of the sphere:

$$\hat n = \frac{\vec r'-\vec r}{|\vec r'-\vec r|} \qquad \text{and} \qquad |\vec r'-\vec r| = R$$

$$\vec\nabla_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right) = -\frac{(\vec r'-\vec r)}{|\vec r-\vec r'|^3}$$

and $\hat n\cdot\vec\nabla_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right) = -\frac{R^2}{R^4} = -\frac{1}{R^2}$

So $\int_V dV'\,\nabla^2_{r'}\left(\frac{1}{|\vec r-\vec r'|}\right) = \oint_{\substack{\text{sphere}\\ S^2}} dS'\left(-\frac{1}{R^2}\right) = 4\pi R^2\left(-\frac{1}{R^2}\right) = -4\pi$

This result is *independent* of the radius of the sphere. The only way that can be true is if

$$\nabla_{r'}^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = 0 \quad \text{for} \quad \vec{r} \neq \vec{r}'$$

(See homework problem #7.)

But the volume integral of the peculiar function $\nabla_{r'}^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$ does *not* vanish so the integrand cannot vanish everywhere. It must be *infinite* at $\vec{r} = \vec{r}'$.

$$\nabla_{r'}^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = C \, \delta^3(\vec{r}-\vec{r}')$$

Let us determine the constant C:

$$\int_V dV' \, \nabla_{r'}^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi = \int_V dV' C \, \delta^3(\vec{r}-\vec{r}') = C$$

$$\underline{\underline{C = -4\pi}}$$

Divergence of $\vec{E}(\vec{r})$

$$\vec{\nabla}_r \cdot \vec{E}(\vec{r}) = -\int_V dV' \rho(\vec{r}\,') \, \nabla^2_{r'} \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right)$$

$$= -\int_V dV' \rho(\vec{r}\,') \left[-4\pi \, \delta^3(\vec{r}-\vec{r}\,') \right]$$

$$= +4\pi \rho(\vec{r})$$

$$\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r})}$$ differential form of Gauss' law

If we substitute the definition $\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$

$$\nabla^2 \Phi(\vec{r}) = -4\pi \rho(\vec{r})$$ is called the <u>Poisson equation</u>

If the region we are considering is charge-free, that is $\rho(\vec{r}) = 0$

then

$$\nabla^2 \Phi(\vec{r}) = 0$$ is the <u>Laplace Equation</u>

Integral form of Gauss' Law:

. Start with $\vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi\rho(\vec{r})$

Integrate both sides over a Volume V

$$\int_V dV\, \vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi\int_V dV\rho(\vec{r}) = 4\pi\, Q_{\text{in } V}$$

$\|$ divergence theorem

$$\boxed{\oint_S dS\ \hat{n}\cdot\vec{E} = 4\pi\ Q_{\substack{\text{enclosed}\\ \text{in } S}}}$$

integral Gauss' Law

You are ready for problem # 8

Next Time:

What is needed to specify $\Phi(\vec{r})$ in a

Finite region if the charge density $\rho(\vec{r})$

is known only in that region (not everywhere)?

Look at Green's Identities

—— End Lecture # 3 ——