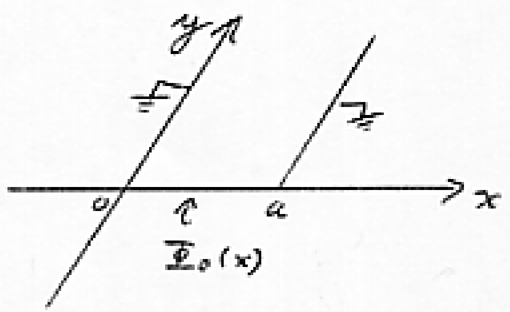
Previously, we considered the problem of finding the potential in the channel $(y>0), (0<x<a)$ when the potential was specified on the finite length $(0<x<a)$.



We extended the boundary condition outside the region of physical interest.

We made the function <u>odd</u>: $\Phi_0(x) = -\Phi_0(-x)$

and <u>periodic</u> with period $2a$: $\Phi_0(x+2a) = \Phi_0(x)$

We found solutions of the form:

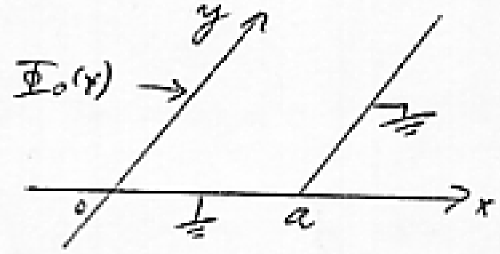
$$\Phi_0(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi n x}{2a}\right) \quad \text{or} \quad \sum_n A_n \sin(k_n x)$$

where k_n is the <u>wavenumber</u>. The more familiar wavelength λ is $\frac{2\pi}{k}$.

We see that only denumerably many wavenumbers are required to build the solution, namely:

$$K_n = \frac{\pi}{a}, \frac{2\pi}{a}, \frac{3\pi}{a}, \ldots$$

Now consider a related problem. Again, we want to determine the potential in the channel $(y > 0), (0 < x < a)$, but this time the boundary condition is specified on the positive y-axis.



We demand that $\Phi_0(y)$ vanish "sufficiently fast" as $y \to \infty$.

Now we need a complete set of functions of the variable y. Remember sines and cosines are complete, but real exponentials are not. We can represent any function in a complete basis.

Clearly, we need a solution of the form

$$\Phi_0(y) = B(k) \sin(ky) + A(k) \cos(ky)$$

but now instead of summing over denumerably many wavenumbers, we must integrate even non-denumerably many k's. Essentially, the reason is that now the "period" of $\Phi_0(y)$ is infinite, or some would say that it is not periodic

$$\Phi_0(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[B(k) \sin(ky) + A(k) \cos(ky) \right]$$

The factor of 2π in the measure above is a normalization.

We are still free to extend $\Phi_0(y)$ outside the range of interest, that is $y < 0$, any way we please. Let's choose to make $\Phi_0(y)$ an odd function:

$$\Phi_0(y) = -\Phi_0(-y)$$

Then the $A(k)$ function vanishes and we do not need the cosines to express the arbitrary function $\Phi_0(y)$.

We use growing and decaying real exponentials for the behavior in the x-direction.

The solution in the channel must be:

$$\Phi(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \; B(k) \sin(ky) \sinh\left[k(a-x)\right]$$

$\sinh(\alpha x)$ and $\cosh(\alpha x)$ are equivalent to $e^{+\alpha x}$ and $e^{-\alpha x}$ in linear combinations; $\sinh(\alpha x) = \frac{1}{2}(e^{\alpha x} - e^{-\alpha x})$

$\cosh(\alpha x) = \frac{1}{2}(e^{\alpha x} + e^{-\alpha x})$

This solution satisfies Laplace's equation in the channel

$$\nabla^2 \Phi(x,y) = 0$$

and satisfies two of the boundary conditions:

$$\Phi(x,0) = 0 \qquad \text{for } 0 < x < a$$

$$\Phi(a,y) = 0 \qquad \text{for } y > 0$$

The last boundary condition: $\Phi(0,y) = \Phi_0(y)$ will determine the coefficient function $B(k)$.

This "Fourier series" with non-denumerably many wavenumbers is called the

Fourier Integral Transform

In general:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[A(k)\cos(kx) + B(k)\sin(kx)\right]$$

$$A(k) = \int_{-\infty}^{\infty} dx\, f(x)\cos(kx) = A(-k) \qquad \underline{\text{even}}$$

$$B(k) = \int_{-\infty}^{\infty} dx\, f(x)\sin(kx) = -B(-k) \qquad \underline{\text{odd}}$$

This lets us write:

$$f(x) = 2\int_{0}^{\infty} \frac{dk}{2\pi}\left[A(k)\cos(kx) + B(k)\sin(kx)\right]$$

Now let's solve for $B(k)$ in our problem

$$\Phi_o(y) = 2\int_0^{\infty} \frac{dk}{2\pi} B(k) \sin(ky) \sinh(ka)$$

$$B(k) \sinh(ka) = \int_{-\infty}^{\infty} dy' \sin(ky') \Phi_o(y')$$

$$= 2\int_0^{\infty} dy' \sin(ky') \Phi_o(y')$$

$$B(k) = \frac{2}{\sinh(ka)} \int_0^{\infty} dy' \sin(ky') \Phi_o(y')$$

Now that we have $B(k)$, we can substitute this back into the solution $\Phi(x,y)$:

$$\Phi(x,y) = \int_0^{\infty} dy' \; K(x,y;y') \, \Phi_o(y')$$

$K(x,y;y')$ is called the <u>kernel</u> and for this problem:

$$K(x,y;y') = \frac{2}{\pi}\int_0^{\infty} dk \; \frac{\sin(ky)\sin(ky')\sinh[k(a-x)]}{\sinh(ka)}$$

$$= \int_0^{\infty} \frac{dk}{\pi}\left[\cos k(y-y') - \cos k(y+y')\right] \frac{\sinh[k(a-x)]}{\sinh(ka)}$$

The dummy variable k can actually be integrated out in this problem, as long as we are careful to distinguish two regions:

$\boxed{x \neq 0}$ $K(x,y;y') =$

$$\frac{1}{2}\left[\frac{\sin\left[\pi\left(1-\frac{x}{a}\right)\right]}{\cos\left[\pi\left(1-\frac{x}{a}\right)\right]\cosh\left[\frac{\pi}{a}(y-y')\right]} - \frac{\sin\left[\pi\left(1-\frac{x}{a}\right)\right]}{\cos\left[\pi\left(1-\frac{x}{a}\right)\right]\cosh\left[\frac{\pi}{a}(y+y')\right]}\right]$$

$\boxed{x = 0}$

$$K(0,y;y') = \delta(y-y')$$

iii) The boundary conditions depend on all three coordinates.

Suppose we want to find the potential inside a parallelopiped (box) with one corner at the origin: $0 < x < a$, $0 < y < b$, $0 < z < c$. Suppose further that the potential is specified on the six sides of the box (Dirichlet boundary conditions).

side #1	$(z = c)$	$\Phi(x,y,c) = f_1(x,y)$
side #2	$(z = 0)$	$\Phi(x,y,0) = f_2(x,y)$
side #3	$(y = b)$	$\Phi(x,b,z) = f_3(x,z)$
side #4	$(y = 0)$	$\Phi(x,0,z) = f_4(x,z)$
side #5	$(x = a)$	$\Phi(a,y,z) = f_5(y,z)$
side #6	$(x = 0)$	$\Phi(0,y,z) = f_6(y,z)$

We assume there are no charges in the box so the solution satisfies Laplace's equation inside V

$$\nabla^2 \Phi(x,y,z) = 0$$

RESERVE

It is very difficult if not impossible to write down a solution which will simultaneously satisfy all 6 boundary conditions. Therefore, we will use the principle of superposition in the following way.

First consider a <u>much</u> easier problem. The potential on side # 1 is still $f_1(x,y)$, but now imagine that the other 5 sides are <u>grounded</u>: $f_2 = 0 = f_3 = f_4 = f_5 = f_6$.

Call the solution to this problem $\Phi_1(x,y,z)$. This can be done quite easily, as will we see.

The plan is to perform this procedure for each of the 6 sides in turn, obtaining partial solutions:

$\Phi_1(x,y,z)$; $\Phi_2(x,y,z)$; ...; $\Phi_6(x,y,z)$

The solution to the original problem is, by superposition

$$\Phi(x,y,z) = \sum_{i=1}^{6} \Phi_i(x,y,z)$$

Now let's see how to solve for <u>one</u> of the partial solutions: $\Phi_1(x,y,z) = X(x)\,Y(y)\,Z(z)$ (separation of variables)

$\Phi_1(x,y,z)$ satisfies Laplace's equation in the box,

$$\nabla^2 \Phi_1(x,y,z) = 0 = X''(x)\,Y(y)\,Z(z) + X(x)\,Y''(y)\,Z(z) + X(x)\,Y(y)\,Z''(z)$$

divide by $X(x)\,Y(y)\,Z(z)$:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

This can only be true if each fraction is a <u>constant</u>.

Now we must be a little circumspect when choosing the constants. Since we want to match the boundary condition of side #1, we will need to expand $f_1(x,y)$ in a <u>complete</u> set of functions of x and y. Therefore, we want sines and cosines of x and y. Then we are <u>forced</u> to have exponentials in z.

In the following, α and β are real and positive.

$$\frac{X''_{(x)}}{X_{(x)}} = -\alpha^2 \implies X_{(x)} = A_1 \sin(\alpha x) + \cancel{A_2}^{0} \cos(\alpha x)$$

$$\frac{Y''_{(y)}}{Y_{(y)}} = -\beta^2 \implies Y_{(y)} = B_1 \sin(\beta y) + \cancel{B_2}^{0} \cos(\beta y)$$

Then since $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$ we are forced to set

$$\frac{Z''_{(z)}}{Z_{(z)}} = +(\alpha^2+\beta^2) \implies Z_{(z)} = C_1 \sinh(\sqrt{\alpha^2+\beta^2}\, z) + \cancel{C_2}^{0} \cosh(\sqrt{\alpha^2+\beta^2}\, z)$$

The coefficients A_2, B_2, and C_2 vanish by the boundary conditions: $f_6 = 0$, $f_4 = 0$, and $f_2 = 0$, respectively.

The boundary condition $f_5 = 0$ implies $\alpha = \frac{n\pi}{a}$

and $f_3 = 0$ implies $\beta = \frac{m\pi}{b}$

Solutions have the form:

$$\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\sqrt{\alpha^2+\beta^2}\; z\right)$$

$$= \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\;\pi z\right)$$

The partial solution $\Phi_1(x,y,z)$ is a linear combination

$$\Phi_1(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi z\right)$$

The solution satisfies Laplace's equation and satisfies 5 of the boundary conditions.

The last one, on side #1, determines A_{nm}.

We will need the orthogonality property of this double Fourier series to extract the coefficients A_{nm}.

$$\int_0^a dx \int_0^b dy \left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\right]\left[\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{b}\right)\right]$$

$$= \frac{ab}{4}\,\delta_{nn'}\,\delta_{mm'}$$

On side #1, where $z = c$:

$$\Phi(x,y,c) = f_1(x,y)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi c\right)$$

Multiply both sides by $\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{b}\right)$

and integrate $\int_0^a dx \int_0^b dy$.

You will find:

$$A_{nm} = \frac{4}{ab}\frac{1}{\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi c\right)}\int_0^a dx\int_0^b dy\; f_1(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m y\pi}{b}\right)$$

The vector analogue of what we are doing may help. Suppose you have an arbitrary vector $\vec{F}$ and you wish to expand it in an orthonormal basis.

$$\vec{F} = \sum_i A_i \hat{e}_i$$

How do you determine the coefficients A_i?

Dot $\hat{e}_j$ into both sides and use $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

$$\hat{e}_j \cdot \vec{F} = \sum_i A_i\, \hat{e}_i \cdot \hat{e}_j = \sum_i A_i \delta_{ij} = A_j$$

$$\boxed{A_i = \vec{F}\cdot\hat{e}_i}$$

End Lecture #10